Preliminary Examination for Real and Complex Analysis-September, 2004

There are 300 total points. The problems in Part I are worth 20 points each, while the problems in Parts II and III are worth 40 points.

Part I

Do all three problems in this section.

(1) Given (Ω, Σ), where Σ is a sigma-algebra of subsets of Ω, define a real-valued measurable function, and prove that the sum of two such functions is measurable.

(2) Suppose a real-valued nonnegative function f is summable on the measure space (Ω, Σ, μ) and

$$\int_{\Omega} f \, d\mu = 0.$$

What do you conclude about f? Give the proof.

(3) Find the Laurent expansion for the function

$$f(z) = \frac{z-1}{z(z-2)^3}$$

in the annulus $\{z : 0 < |z-2| < 2\}$.

Part II

Do any four of the problems in this section. If you do more than four, indicate which ones should be graded.

(1) Prove the following theorem.

Theorem. A normed linear space X is complete if, whenever $\{x_n\} \subset X$ and $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\sum_{n=1}^{\infty} x_n \in X$, that is, if every absolutely convergent series is convergent.

Use this theorem directly to prove the completeness of $L^p(\Omega), 1 \leq p < \infty$.

(2) Define the translation operator

$$(\tau_h f)(x) = f(x-h), \;\; h, x \in \mathbb{R}^n.$$

Prove that τ_h is continuous on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. State explicitly what property of the continuous compact support functions and what property of Lebesgue measure you are using in the course of the proof.

(3) Give a counter-example to show that a closed operator need not be continuous. Then state and prove the closed graph theorem.

(4) Let (Ω, Σ, μ) be a measure space. Show that this space has a completion $(\Omega, \bar{\Sigma}, \bar{\mu})$, defined as follows.

$$\bar{\Sigma} = \{E \cup A : E \in \Sigma, \; A \subset B \text{ for some } B \in \Sigma \text{ such that } \mu(B) = 0\},$$

$$\bar{\mu}(E \cup A) = \mu(E).$$

(5) Let X be a complete metric space. Prove that any countable collection of dense open subsets of X has nonempty intersection.

(6) Determine the Fourier transform of

$$g(x) = \exp(-\pi |x|^2), \;\; x \in \mathbb{R}^n.$$

Part III

Do any two problems in this section. If you do more than two, indicate which ones should be graded.

(1) Find all linear fractional transformations ϕ that map the upper half-plane onto the disk $D = \{w : |w| < R\}$.

(2) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}, \qquad -1 < a < 1.$$

(3) How many zeros does $\sin(z) + 2iz^2$ have inside the rectangle,

$$\{z : \ |\mathrm{Re}(z)| < \frac{\pi}{2}, \ |\mathrm{Im}(z)| \leq 1\}?$$

Justify your answer.