Preliminary Examination in Analysis **September 2005**

Part 1 Do all three problems in this part.

Problem 1. (a) Describe the major steps (without proof) of constructing the Lebesgue measure on the real line. (b) Show that the Lebesgue measure is complete, i.e., a subset of a measurable set of measure zero is always measurable. (c) Show that a countable subset of the real line has Lebesgue measure zero and give an example of an uncountable subset which also has measure zero.

Problem 2. (a) State the three limit theorems of Lebesgue integration theory (monotone convergence theorem, Fatou's lemma and dominated convergence theorem). (b) Using Fatou's lemma, or otherwise, prove the dominated convergence theorem.

Problem 3. Use complex integration theory to compute the following two integrals:

$$(a)\ \int_{-\infty}^{\infty} \frac{dx}{1+x^4} \qquad (b)\ \int_{0}^{\infty} \frac{x^{1-\alpha}}{1+x^2}dx, \quad 0<\alpha<2.$$

Part 2 Choose one problem in this part.

Problem 4. (a) Define the product measure space of two measure spaces and state the Fubini theorem. (b) Show that if $f,\ g \in L^1(R)$ then the convolution

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$$

is well defined and

$$\|f*g\|_1 \le \|f\|_1\,\|g\|_1.$$

Problem 5. (a) State Hölder's inequality. (b) Prove the inequality

$$\|fgh\|_1 \le \|f\|_\alpha \|g\|_\beta \|h\|_\gamma, \qquad \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=1.$$

Problem 6. State and outline a proof of the Hahn-Banach theorem.

Part 3 Choose one problem in this part.

Problem 7. Let $(X,\mathcal{F},\mu)$ be a measure space. Suppose that $f\in L^p(X,\mathcal{F},\mu)$ for all $p>0$. Show that $\phi(p)=\|f\|_p$ is a continuous function of p on $(0,\infty)$.

Problem 8. Let f be a real-valued integrable function on $[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} e^{int} f(t)\,dt = 0$$

for all integers n except $n = 0,\ 1,\ -1$. Show that there are real numbers c_0, c_1 and c_2 such that $f(t) = c_0 + c_1 \sin t + c_2 \cos t$.

Problem 9. Let f be a nonnegative measurable function on a measure space $(X, \mathcal{F}, m)$. Show that

$$\int_X f\,dm = \int_0^\infty m(f > \lambda)\,d\lambda.$$

Part 4 Do the single problem in this part.

Problem 10. Let $\Omega = \{z = x + iy :\ y \neq 0\}$ and $f \in C(R) \cap L^1(R)$. Define

$$u(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - z} d\xi.$$

(a) Prove that u is holomorphic on Ω and $\lim_{|y|\to\infty} u(z) = 0$.

(b) Compute $\lim_{y\to 0} \{u(z) - u(\overline{z})\}$.

(c) If $f \in C(R) \cap L^1(R) \cap L^p(R)$, $p > 1$, prove that the limit in (b) takes place in $L^p(R)$.

Part 5 Choose one problem in this part.

Problem 11. Let X be a Banach space. Show that X is locally compact if and only if it is finite dimensional.

Problem 12. Let f be a holomorphic function on the entire complex plane C which is one-to-one and onto (i.e., $f(C) = C$). Show that it has the form $f(z) = a_0 + a_1 z$ with $a_1 \neq 0$.

Problem 13. Let $f \in L^1[a, b]$ and

$$F(x) = \int_a^x f(y)\,dy, \qquad a \le x \le b.$$

Show that F has bounded variation on $[a, b]$ and that its total variation on $[a, b]$ is given by the formula

$$V(F)_a^b = \int_a^b |f(y)|\,dy.$$