Preliminary Exam in Analysis (Fall, 2006)

Do all problems.

(1) Let $p \in [1, \infty]$ and let g be a measurable function on $\mathbb{R}$ such that $fg \in L^1(\mathbb{R})$ for all $f \in L^p(\mathbb{R})$. Show that $g \in L^q(\mathbb{R})$ (with q the conjugate exponent to p).

(2) Let $f \in L^p([0,1])$ for all $p \in [1, \infty]$. Show that

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

(3) Let f and g be two entire functions such that, for all $z \in \mathbb{C}$, $\operatorname{Re} f(z) \leq k \operatorname{Re} g(z)$ for some real constant k (independent of z). Show that there are constants a, b such that

$$f(z) = ag(z) + b.$$

(4) Show that every biholomorphic map of the unit disc onto itself is given by a linear fractional transformation.

(5) Given a sequence $\{a_n\}$ dense in $\mathbb{R}$ and a set F of positive measure, show that the set

$$G = \bigcup_{n=1}^{\infty} (F + a_n)$$

is all of $\mathbb{R}$ except for a set of measure zero.

(6) (a) Prove that if $f \in L^1(\mathbb{R})$, and $\hat{f}$ denotes its Fourier transform, then $\hat{f} \in \mathcal{C}(\mathbb{R})$, with $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.

(b) Show that the conditions on $\hat{f}$ in the first part of the problem are *not sufficient* to ensure $f \in L^1(\mathbb{R})$. (Hint: show that an estimate would have to hold, if this were so; then show the estimate is false by considering the family of functions $1_{[-A,A]}(x)$.)