Preliminary Examination in Analysis

September 10th, 2007

Instruction: Do all problems.

PROBLEM 1. State the following convergence theorems in Lebesgue integration theory:
(1) Fatou's lemma;
(2) Monotone convergence theorem;
(3) Dominated convergence theorem.

PROBLEM 2. Let $(X, \mathscr{F}, \mu)$ be a measure space such that $\mu(X) < \infty$. A sequence of measurable functions $\{f_n\}$ is called uniformly integrable if

$$\lim_{C\to\infty} \sup_n \int_{\{|f_n|\geq\, C\}} |f_n| \, d\mu = 0.$$

Use the convergence theorems in PROBLEM 1 to show that if $\{f_n\}$ is uniformly integrable and $f_n \to f$ almost everywhere, then $f_n \to f$ in $L^1(X, \mathscr{F}, \mu)$.

PROBLEM 3. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4 + 1} \, dx.$$

PROBLEM 4. Let $f \in L^1[a, b]$ and

$$F(y) = \int_a^y f(x) \, dx, \qquad a \leq x \leq b.$$

Show that F has bounded variation on $[a, b]$ and the total variation is given by

$$V(F)_a^b = \int_a^b |f(x)| \, dx.$$

PROBLEM 5. Let $\{e_n,\ n \geq 1\}$ be a complete orthonormal system on a Hilbert space H. Let $\{f_n,\ n \geq 1\}$ be an orthonormal system such that

$$\sum_{n=1}^{\infty} \|f_n - e_n\|^2 < 1.$$

Show that $\{f_n,\ n \geq 1\}$ is also a complete orthonormal system on H.

PROBLEM 6. Let $g : \Delta \to \mathbb{R}$ be a real-valued function on the open disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with the following properties:

(1) $0 \le g(z) \le |z|^2$ for all $z \in \Delta$;
(2) There exists a holomorphic function $f : \Delta \to \mathbb{C}$ such that $|f(z)| = e^{g(z)}$.

Prove that g vanishes identically on Δ.

PROBLEM 7. For an integrable function $f \in C(\mathbb{R})$ on the real line $\mathbb{R} = (-\infty, \infty)$, its Fourier transform $\hat{f}$ is defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx.$$

Show that if $f, g \in L^1(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(t) g(t)\, dt.$$

PROBLEM 8. State the Hahn-Banach theorem and use it to prove the following separation theorem: if X_0 is a closed subspace of a Banach space and $x_0 \in X \backslash X_0$. Then there is a bounded linear functional f on X such that $f(x) = 0$ for $x \in X_0$ and $f(x_0) = 1$.

PROBLEM 9. Prove that there is no holomorphic bijection between the punctured disk $0 < |z| < 1$ and the annulus $1 < |z| < 2$ in $\mathbb{C}$.