Preliminary Examination in Analysis
September 18, 2009

PART I. Do any **three** of the four problems in this part.

Problem I.1. Prove that if $X \subset [0,1]$ is Borel measurable, then for any $\epsilon > 0$ there is an open set U containing X such that $\mu(U \setminus X) < \epsilon$.

Problem I.2. Suppose $\{f_n\}$ is a sequence of measurable functions and $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere. Prove that f is measurable.

Problem I.3. Prove or give a counterexample: If $\{f_n\}$ is a sequence of measurable functions defined on $[0,1]$ such that $0 \le f_n(x) \le 1$ and

$$\lim_{n\to\infty} \int f_n \, d\mu = 0,$$

then for almost all $x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = 0.$$

Problem I.4. State the following three convergence theorems in the Lebesgue integration theory and outline a proof of one of them.
(1) Fatou's lemma;
(2) Monotone convergence theorem;
(3) Dominated convergence theorem.

PART II Do any **four** of the five problems in this part.

Problem II.1. Prove that a linear function from a Hilbert space H to itself is continuous if H is finite dimensional. Give an example of a linear function from a subspace of a Hilbert space to itself which is not continuous.

Problem II.2. Define the function ν from subsets of $\mathbb{R}$ to $[0,\infty]$ by $\nu(A) = \infty$ if 0 is in the closure of A and $\nu(A) = 0$ otherwise. Prove that ν is finitely additive but not countably additive.

Problem II.3. Let $(X, \mathcal{B}, \mu)$ be a measure space, and that $f, g \in L^p(X, \mathcal{B}, \mu)$ have positive L^p-norm. Suppose that $\|f+g\|_p = \|f\|_p + \|g\|_p$. Show that

$$\frac{f}{\|f\|_p} = \frac{g}{\|g\|_p}, \qquad \mu - \text{a.e.}$$

Problem II.4. State the Baire Category Theorem. Is it possible for a sequence of continuous functions $f_n\colon [0,1] \to [0,1]$ to have a pointwise limiting function $f\colon [0,1] \to [0,1]$ that is 0 on the rationals and 1 on the irrationals? Give an example or prove it is not possible.

Problem II.5. Suppose that X and Y are compact Hausdorff spaces and $f\colon X \times Y \to \mathbf{R}$ is a continuous function. Prove that for every $\epsilon > 0$ there exist $n > 0$ and continuous functions $f_1, f_2, \ldots, f_n$ on X and continuous functions $g_1, g_2, \ldots, g_n$ on Y such that:

$$\|f - \sum_{i=1}^{n} f_i g_i\|_\infty < \epsilon,$$

where $\| \ \|_\infty$ denotes the sup norm.

PART III Do any **three** of the four problems in this part.

Problem III.1. Let $f(z) = z^4 + \frac{z^3}{4} - \frac{1}{4}$ How many zeros does f have in $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$?

Problem III.2. Let A be the set of $z \in \mathbb{C}$ such that $|z| \leq 1$, $\mathrm{Im}(z) \leq 0$, and $z \notin \{1, -1\}$. Find an explicit continuous function $u\colon A \to \mathbb{R}$ such that

- u is harmonic on the interior of A,
- $u(z) = 3$ for $z \in A \cap \mathbb{R}$
- $u(z) = 7$ for z in the intersection of A with the unit circle.

Problem III.3. Find explicitly a Riemann map (that is, a biholomophic bijection) of the open unit disk D onto each of the following domains:

(1) the whole plane minus the nonpositive real axis;
(2) the first quadrant;
(3) the intersection of the unit disk with the upper half plane;
(4) the unit disk minus the segment $[0, 1)$.

Problem III.4. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is real analytic and periodic with period 2π. Prove that f has an analytic continuation F defined on a strip

$$S = \{x + iy \in \mathbb{C} : |y| < \rho\}$$

with $\rho > 0$, and that $F(z + 2\pi) = F(z)$ for $z \in S$.