ANALYSIS PRELIMINARY EXAM
FRIDAY, SEPTEMBER 16, 2011

Part I. Do four of the following five problems.

(1) (a) State the monotone convergence theorem, the dominated convergence theorem and Fatou's lemma.
(b) Show that the monotone convergence theorem can fail for a sequence of not necessarily nonnegative functions.
(c) Use the monotone convergence theorem to prove Fatou's lemma.

(2) State Hölder's inequality (including the condition for the case of equality) and use it to prove the following inequality: for positive α, β, and γ and measurable functions f, g, and h on a measure space,

$$\|fgh\|_1 \le \|f\|_\alpha \|g\|_\beta \|h\|_\gamma, \qquad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

(3) Suppose that $f \in L^1[0,1]$ and $\lambda > 0$. Show that the integral

$$F_\lambda(x) = \int_0^x (x-t)^{\lambda-1} f(t)\, dt$$

exists for almost every $x \in [0,1]$ and $F_\lambda \in L^1[0,1]$.

(4) Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ and $V(f)_a^b$ the total variation of f on the interval $[a,b]$. Show that

$$\int_a^b |f'(t)|\, dt \le V(f)_a^b.$$

(5) We use $\mu(f)$ to denote the integral of a function f with respect to a measure μ. Let $\{\mu_n,\ n \ge 0\}$ be a sequence of Borel measures on $[0,1]$ such that $\mu_n(f) \to \mu_0(f)$ for all continuous function f on $[0,1]$. Show that

$$\mu_0(O) \le \liminf_{n\to\infty} \mu_n(O)$$

for every open set $O \subset [0,1]$.

Part II. Do two of the following four problems.

(1) Suppose that Y is a finite dimensional (in the usual algebraic sense) subspace of a Banach space X. If $x_n \in Y$ and $x_n \to x$ in X, then $x \in Y$; namely, every finite dimensional subspace of a Banach space is closed.

(2) Let X be a Banach space. A linear operator $T : X \to X$ is compact if the image of every bounded set is precompact (i.e., the closure is compact). Let $K : [0,1]^2 \to \mathbb{R}$ be a continuous function on the unit square. Show that the integral operator

$$Kf(x) = \int_0^1 K(x,y) f(y)\, dy$$

is compact on $C[0,1]$.

(3) Let $\{e_n\}$ be an orthonormal basis for a Hilbert space H. Let $\{f_n\}$ be an orthonormal set in H such that

$$\sum_{n=1}^{\infty} \|f_n - e_n\| < \infty.$$

Show that $\{f_n\}$ is also an orthonormal basis for H.

(4) (a) State the Open Mapping Theorem and the Closed Graph Theorem for Banach spaces.
(b) Let X be a linear vector space that is complete in the norms $|\cdot|$ and $\|\cdot\|$. Prove that if there is a constant C such that $|x| \leq C\|x\|$ for all $x \in X$, then there is another constant C_1 such that $\|x\| \leq C_1|x|$.

Part III. Do four of the following five problems.

(1) Let $\xi \in \mathbb{R}$. Use the method of residues to calculate the integral

$$\int_{\mathbb{R}} \frac{e^{-2\pi i x\xi}}{\cosh \pi x}\, dx.$$

(2) Find the number of roots of $z^7 - 5z^4 + 8z - 1 = 0$ in the annulus $\{1 < |z| < 2\}$.

(3) Consider the power series

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

(a) Find the radius of convergence of the power series.
(b) Find the maximal open subset of $\mathbb{C}$ to which the power series can be analytically continued.

(4) (a) Suppose that f is holomorphc on a disk $D(0;R) = \{|z| \leq R\}$ and satisfies the bound $|f(z)| \leq M$ on $|z| = R$. Show that

$$|f(z) - f(0)| \leq \frac{2M|z|}{R}.$$

(b) State Liouville'e theorem and use part (a) to give a proof of it.

(5) Let $\alpha \in \mathbb{C}$ such that $|\alpha| < 1$ and let

$$L(z) = \phi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}.$$

Let $L_1 = L$ and $L_{n+1} = L \circ L_n$. Show that $\lim_{n\to\infty} L_k$ exists uniformly on compact subsets of the unit disk $D(0,1)$ and determine the limit function. Hint: show that L_n is a Möbius transformation ϕ_{α_n} with $|\alpha_n| < 1$ and find $\lim_{kn\to\infty} \alpha_n$.