Department of Mathematics, Northwestern University

Preliminary Examinations in Analysis, Friday September 17, 1999

1. Let $\{X, \mathcal{A}, \mu\}$ be a σ–finite measure space and let $E \in \mathcal{A}$. Prove that for every measurable function $f : E \to I\!R$ and all $p > 0$,

$$\int_E |f|^p \, d\mu = p \int_0^\infty t^{p-1} \, \mu([|f| > t]) \, dt.$$

2. Let $\{X, \mathcal{A}, \mu\}$ and $\{Y, \mathcal{B}, \nu\}$ be two complete measure spaces and let $f \in L^p(X \times Y)$ for some $p \geq 1$. Prove that,

$$\left(\int_X \left| \int_Y f(x,y) \, d\nu \right|^p d\mu \right)^{1/p} \leq \int_Y \|f(\cdot, y)\|_{p,X} \, d\nu.$$

3. Let $\{f_n\} \to f$ weakly in $L^p(E)$ for $p \in (1, \infty)$, and assume that $\|f_n\|_p \to \alpha$ for some $\alpha > 0$. If $\|f\|_p = \alpha$, then $\{f_n\} \to f$ strongly in $L^p(E)$. Prove this statement for $p = 2$. If $\|f\|_p \neq \alpha$, then the conclusion is false. Give a counterexample.

4. Let $\{X, \mathcal{A}, \mu\}$ be a measure space and let $E \in \mathcal{A}$ be of finite measure. Prove that almost everywhere convergence in E implies convergence in measure. This is false if E is not of finite measure. Give a counterexample.

5. The Legendre transform f^* of a convex function $f : I\!R^N \to I\!R$ is defined by,

$$f^*(x) = \sup_{y \in I\!R^N} \{x \cdot y - f(y)\}.$$

Prove that f^* is convex.

6. Let $f : I\!R^N \to I\!R$ be convex and satisfying the growth condition,

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty.$$

Prove that f^* satisfies the same growth condition as $|x| \to \infty$.