Preliminary Examination in Real Analysis **September 1995**

Instructions: You have **three** hours. You need to work out totally **six** problems, **four** in part I, and **two** in part II.

PART I

1. On R^n, state Fatou's Lemma and use it to prove the Lebesgue Dominated Convergence Theorem.

2. Let E be a Lebesgue measurable set in R^n. Define

   a. Convergence a.e. on E.

   b. Convergence in measure on E.

   c. Uniform convergence on E.

Write down all the possible relations among a. b. and c.

   d. Define (strong) convergence in $L^p(E)$, $1 \leq p < \infty$.

Does d. imply b.? Does b. imply d.? If your answer is YES, give a proof. If your answer is NO, give a counterexample.

3. Let $(X, \mathcal{B})$ be a measurable space and let ν and μ be σ-finite measures on $(X, \mathcal{B})$.

   a. Define $\nu \perp \mu$ and $\nu \ll \mu$.

   b. State the Radon-Nikodym Theorem.

   c. Extend the Radon-Nikodym Theorem to the case of signed measure and prove it.

Instructions: Work either problem 4. or 5. Indicate which problem you want to be graded.

4. State the Baire Category Theorem and use it to prove the Uniform Boundedness Principle.

5. State and prove the Riesz Representation Theorem for $L^p(E)$, where $1 \leq p < \infty$ and $E \subset R^n$ is a Lebesgue measurable set of finite measure.

PART II

Instructions: Work TWO of the following FOUR problems. Indicate which two problems you want graded.

6. Let $K(x,y) \in L^q(R^2)$ and $f(y) \in L^p(R^1)$, where $1 \le q < \infty$, $1 < p \le \infty$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Show that the operator T defined by

$$Tf(x) = \int K(x,y) f(y)\, dy$$

is a bounded linear operator from $L^p(R^1)$ to $L^q(R^1)$.

7. Suppose that $(X, \mathcal{B})$ is a measurable space and ν and μ are finite measures on $(X, \mathcal{B})$, such that $\nu \ll \mu$. If $\lambda = \nu + \mu$ and if $f = \left[\dfrac{d\nu}{d\lambda}\right]$, show that $0 \le f(x) < 1$ a.e. λ.

8. Let f_n be a sequence of functions in $L^p(E)$ and g_n be a sequence of functions in $L^q(E)$, where $E \subset R^n$ is a Lebesgue measurable set, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $1 < p < \infty$. If $f_n \to f$ strongly in $L^p(E)$ and $g_n \rightharpoonup g$ weakly in $L^q(E)$, show that $\int f_n g_n \to \int fg$.

9. Let X, Y be normed linear spaces, and X^*, Y^* denote their conjugate spaces. On $X \times Y$, we define

$$\|(x,y)\| = (\|x\|^2 + \|y\|^2)^{1/2}.$$

Show that for any $F \in (X \times Y)^*$, there is a unique pair of functionals $f \in X^*$, $g \in Y^*$, such that $F(x,y) = f(x) + g(y)$. Moreover, if we define on $X^* \times Y^*$,

$$\|(f,g)\| = (\|f\|^2 + \|g\|^2)^{1/2},$$

show that $(X \times Y)^*$ is isometrically isomorphic to $X^* \times Y^*$.