Preliminary Examination *Feb.15, 1996*

Real Analysis

Do all six problems

Problem 1. State and prove Fatou's lemma.

Problem 2. (a) State the definitions of functions of bounded variations and absolutely continuous functions. (b) Show that every absolutely continuous function is of bounded variation.

Problem 3. (a) State the definition of a measurable subset of R^1. (b) Show that if E is a measurable set of finite measure, then for any $\epsilon > 0$ there exist an open set G and a closed set F such that $F \subset E \subset G$ and $m(G \backslash F) < \epsilon$.

Problem 4. Let f be an integrable function on the measure space $(X, \mathcal{B}, \mu)$. Show that given $\epsilon > 0$, there is a $\delta > 0$ such that for each measurable set E with $mE < \delta$ we have

$$\left| \int_E f d\mu \right| < \epsilon.$$

Problem 5. Let $(X, \mathcal{B}, \mu)$ be a complete measure space and $f \in L^p(\mu)$, $1 \leq p < \infty$. Prove that if $\epsilon > 0$, then there is a simple function ϕ vanishing outside a set of finite measure such that $\|f - \phi\| < \epsilon$.

Problem 6. Let h and g be integrable functions on the complete measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, respectively, and define $f(x, y) = h(x)g(y)$. Prove that f is integrable on the product measure space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$ and

$$\int_{X \times Y} f d(\mu \times \nu) = \left(\int_X d\mu \right) \left(\int_Y g d\nu \right).$$