Preliminary Examination September 15, 1997

Real and Complex Analysis

PART A. *Do problems 1, 2 and 3*

1. (a) Define

$$f(t) = \int_0^{100} \frac{\cos(1/x + t)}{|x + t|^{1/2}}\, dx.$$

Prove or disprove: $\lim_{t\to 0^+} f(t) = f(0)$.

(b) Define

$$g(t) = \int_0^{100} \frac{\cos^{-1/2}(x)}{|x + t^{1/2}|}\, dx.$$

Prove or disprove: $\lim_{t\to 0^+} g(t) = g(0)$.

The notation $t \to 0^+$ indicates limit from the right (the positive t-axis).

2. Prove or give a counterexample for each of the following statements. The functions $f, f_1, f_2, \ldots$ are assumed to be measurable.

(a) The unit ball in $L^1([0,1])$ is compact.

(b) If $f_i \to f$ in $L^2(\mathbf{R}^n)$, then $f_i \to f$ a.e.

(c) Suppose $f, f_1, f_2, \ldots$ possess a common bound in $L^2(\mathbf{R}^n)$ and in $L^\infty(\mathbf{R}^n)$. If $f_i \to f$ a.e., then $f_i \to f$ in $L^2(\mathbf{R}^n)$.

(d) Suppose $f, f_1, f_2, \ldots$ possess a common L^∞ bound in B_1, the unit ball in $\mathbf{R}^n$. If $f_i \to f$ in measure in B_1, then $f_i \to f$ in $L^2(B_1)$.

3. Use the method of residues to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\, dx.$$

PART B. *Do problem 4 or 5*

4. Suppose that $h, h_1, h_2, \ldots$ are L^2 functions on $\mathbf{R}^n$ such that for every continuous function ϕ of compact support,

$$\lim_{i\to\infty} \int_{\mathbf{R}^n} \phi(x) h_i(x)\, dx = \int_{\mathbf{R}^n} \phi(x) h(x)\, dx,$$

Prove that

$$\int_{\mathbf{R}^n} |h(x)|^2\, dx \le \varliminf_{i\to\infty} \int_{\mathbf{R}^n} |h_i(x)|^2\, dx.$$

5. Prove that if u is a function in $L^2([0, 2\pi])$, then

$$\lim_{n\to\infty} \int_0^{2\pi} u(x) \sin(nx)\, dx = 0.$$

PART C. *Do problem 6 or 7*

6. Let B_1 be the ball of radius 1 in $\mathbf{R}^n$. Suppose that f is simultaneously in $L^p(B_1)$ for each $p \in [12, \infty]$. Prove that

$$\lim_{p\to\infty} \|f\|_{L^p(B_1)} = \|f\|_{L^\infty(B_1)}.$$

7. Let f be an L^1 function on $\mathbf{R}^n$. Prove that for a.e. x in $\mathbf{R}^n$,

$$\lim_{r\to\infty} \frac{1}{|B_r|} \int_{B_r(x)} |f(y) - f(x)|\, dy = 0,$$

where $B_r(x)$ be the ball in $\mathbf{R}^n$ with center x and radius r, and $B_r = B_r(0)$. (You are allowed to assume the weak L^1 estimate on the maximal function.)

PART D. *Do problem 8 or 9*

8. How many zeros does $z^6 + 4z^2 - 1$ have in the annulus $\{z : 1 \le |z| \le 2\}$?

9. (a) Find a linear fractional transformation that maps 0 to i, 1 to ∞, and -1 to 1.
(b) Suppose $f(z) \in H(\Omega)$, Ω contains the closed unit disc, $f(0) = 1/2$, and $|f(z)| > 1$ when $|z| = 1$. Prove that $f(z)$ has a zero in the unit disc.