D13 Preliminary Exam

February, 1997

Do all of the problems. Write your solutions in the blue book(s); no notes, books, etc **Notation** Throughout this exam $U = \{z : |z| < 1\}$. A *domain* is an open connected subset of the complex plane. A function is *univalent* if it is one-to-one.

1. Define what it means for an entire function to have finite order; finite genus, finite rank. Give an example of an entire function of infinite order and another example of a function of order 3.

2. Find **all** entire functions f of finite order such that $f(\log n) = n, n = 1, 2, 3, ...$ and prove you are correct.

3. Let D be a domain in the complex plane. (a) Define "normal family" of functions on D (b) State Montel's theorem on normal families of analytic functions on D (c) Let $\mathcal{F}$ be the family of all functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ satisfying $\sum_{k=0}^{\infty} |a_k| \leq 1$.(i)Prove that each $f \in \mathcal{F}$ is analytic on U (ii) Prove that $\mathcal{F}$ is a normal family in U.

4. (a) State Runge's Theorem (b) Use Runge's Theorem (or some other correct procedure) to construct a sequence $\{p_n\}$ of polynomials with $p_n(z) \to 0, \text{Im } z \geq 0$ and $p_n(z) \to 1, \text{Im } z < 0$. (c) Can the polynomials $\{p_n\}$ in (b) be chosen so that $|p_n(re^{i\theta})| \leq C(r), 0 \leq \theta \leq 2\pi, 0 < r < \infty$, where $C(r)$ is a constant that depends on r but not on n or θ? Why or why not? Justify your answer.

5. Is there a univalent analytic function with domain $D_1 = \{z : 0 < |z| < 1\}$ and range
   $D_2 = \{z : \frac{1}{2} < |z| < 1\}$? Why or why not? Prove your assertion.

6. Evaluate these integrals using the residue theorem; justify your steps.

$$(a) \int_{-\infty}^{\infty} \frac{x^2}{(x^2+16)^2(x^2+9)} dx \qquad (b) \int_0^{2\pi} \frac{1}{a+\cos t} dt, a > 1$$

7. Is there an entire function h that satisfies $|h(z)| \geq e^{A|z|}$ for some $A > 0$ and all sufficiently large $|z|$? Give an example of such a function or prove that none exists.

8. Find the Laurent series for the fuunction $f(z) = \frac{z-1}{z(z-2)^2}$ in the region $0 < |z-2| < 2$.

9. How many zeros does $g(z) = z^4 - 3z + 1$ have in U? Justify your answer.

10. Find the explicit form of all univalent analytic functions that map the upper half-plane $\{x + iy : y > 0\}$ onto U.