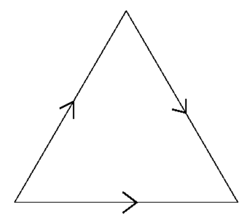
Preliminary Examination in Geometry and Topology, Spring 2015

Do all of the problems. Full credit requires proving that your answer is correct.

1. Let X be the topological space obtained from the standard 2-simplex by identifying all of its three sides as shown:



Calculate $H_k(X;\mathbb{Z})$ and $H^k(X;\mathbb{Z})$, for all k.

2. The Klein bottle M is the surface obtained by taking the square

$$[0,1]\times[0,1]$$

and identifying the vertical sides $(0,t)\sim(1,t)$ to form a cylinder, and the horizontal sides with a reversal of direction $(s,0)\sim(1-s,1)$.

Using the Van Kampen Theorem, give generators and relations for the fundamental group $\pi_1(M)$ of the Klein bottle.

Write down a covering map from the torus $S^1\times S^1$ to the Klein bottle. Give explicit generators for the image under this map of the fundamental group $\pi_1(S^1\times S^1)$ in $\pi_1(M)$. What is the index of this subgroup?

3. Let $n>0$. Consider the standard covering of $\mathbb{RP}^n$ by open sets $\{U_\alpha\}_{0\le\alpha\le n}$, where

$$U_\alpha=\{[x_0:\ldots:x_n]\in\mathbb{RP}^n\mid x_\alpha\neq 0\},$$

and for each $d\in\mathbb{Z}$ define a line bundle E_d on $\mathbb{RP}^n$ by the transition functions

$$g_{\alpha\beta}:U_\alpha\cap U_\beta\to\mathbb{R}\backslash\{0\},\quad g_{\alpha\beta}=\left(\frac{x_\beta}{x_\alpha}\right)^d.$$

a) If a real line bundle E has transition functions $g_{\alpha\beta}$, what are the transition functions of the dual line bundle E^*? Justify your answer.

b) Let $d>0$. Show that $E_d\cong E_1\otimes\cdots\otimes E_1$ (d times), and that $E_{-d}\cong(E_1\otimes\cdots\otimes E_1)^*$,

c) Show that E_d is isomorphic to the trivial bundle if d is even.

d) Show that E_d is isomorphic to E_1 if d is odd.

e) Show that E_1 is not isomorphic to the trivial bundle.

4. Consider the Riemannian manifold

$$M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with metric

$$g = y^{-2}(dx \otimes dx + dy \otimes dy).$$

The Christoffel symbols of the Levi-Civita connection of g (you do **not** need to compute them) are given, in obvious notation, by

$$\Gamma^x_{xx} = 0 \qquad \Gamma^x_{yy} = 0 \qquad \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}$$

$$\Gamma^y_{xx} = \frac{1}{y} \qquad \Gamma^y_{yy} = -\frac{1}{y} \qquad \Gamma^y_{xy} = \Gamma^y_{yx} = 0.$$

a) Write down the equations for a path $t \mapsto (x(t), y(t))$ in M to be a geodesic.

b) Find the equation of the unit-speed geodesic starting at $(0, 1)$ with initial velocity a nonzero multiple of $(0, 1)$.

c) What is the distance in M between the points $(0, a)$ and $(0, b)$?

5. Let $F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3$ be a smooth vector field satisfying the equation $\operatorname{div} F = 0$.

Show that there exist a real number λ and a smooth vector field G on $\mathbb{R}^3 \backslash \{0\}$ such that

$$F = \frac{\lambda}{\rho^3} X + \operatorname{curl} G,$$

where ρ is the distance to the origin $0 \in \mathbb{R}^3$ and X is the tautological vector field of Question 5.

6. A **rotation** of the sphere S^n is the restriction of the action of a matrix in $\mathrm{SO}(n+1)$ to the unit sphere $S^n \subset \mathbb{R}^{n+1}$.

a) Show that every rotation of the even-dimensional sphere S^{2m} has a fixed point.

b) Does every rotation of the 3-dimensional sphere S^3 have a fixed point? If yes, give a proof. If no, find a counterexample.