Preliminary Exam - Differential Geometry (D41/D43, Fall 2000)

Do any **SIX** problems. Mark on your blue book which problem you are omitting. You may use earlier parts of a problem in the later parts.

Problem 1. Let N be a compact manifold of positive dimension. Let $x_0 \in N$ be a basepoint. Show that the two inclusions $i_1 : N \to N \times N, i_1(x) = (x, x)$ and $i_2 : N \to N \times N, i_2(x) = (x_0, x)$ are not homotopic to each other.

Problem 2. Let $f : S^n \to S^n$ be a local diffeomorphism and assume $n > 1$. Show f is a diffeomorphism. Give an example to show that this result is false for $n = 1$.

Problem 3. (a) Define the *Gauss map* for a surface M^2 in $\mathbb{R}^3$.

(b) Consider the surface $z = 6xy - x^2 - y^2$ in $\mathbb{R}^3$. At the point $(0, 0, 0)$ the vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ form an orthonormal basis of the tangent space to this surface. Find the matrix of the derivative of the Gauss map with respect to this basis. Also find the Gaussian curvature, the principal curvatures and the principal directions.

Problem 4. Show that any Riemannian manifold has a unique connection that is symmetric and compatible with the metric.

Problem 5. Consider the upper half plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian metric such that the length of a vector based at the point (x, y) is $1/y$ times the usual Euclidean length of the vector.

(a) Show that vertical lines (i.e. those along which the x-coordinate is constant) are geodesics.

(b) Show that the curvature of the metric is -1 everywhere.

Problem 6. (a) Define differential *k-form.* and what it means for a form to be *closed* and to be *exact*.

(b) Prove that any closed 1-form on $\mathbb{R}^2$ is exact.

(c) Give an example of a 1-form on the two-dimensional torus that is closed but not exact.

Problem 7. State and prove the Hopf-Rinow theorem.