MATH 443 Preliminary Exam Fall 2003

SHOW ALL YOUR WORK!

1. a) State the Inverse Function Theorem.

b) Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$H(x, y) = (ye^x, 2x).$$

Find the set of points at which H is regular. Is H a diffeomorphism, a local diffeomorphism or neither?

2. Let X, Y be two vector fields on a smooth manifold M. The bracket $[X, Y]$ of X and Y is defined by

$$[X, Y]f = X(Yf) - Y(Xf), \qquad f \in C^\infty(M).$$

Show that the set of all vector fields on M is a Lie Algebra over $\mathbb{R}$ with respect to this bracket operation.

3. a) Define a differential k-form in $\mathbb{R}^n$.

b) Consider the 1-form

$$\omega = x^2 dy - y dx + 2dz$$

on $\mathbb{R}^3$. Calculate its exterior derivative $d\omega$.

c) Show that the composition of the exterior differentiation with itself is zero, i.e., $d^2\eta = 0$ for every differential form η.

4. a) State the Cartan Formula which relates the Lie derivative to interior product i and exterior derivative d.

b) Prove that the Lie derivative satisfies the following property for vector fields X, Y:

$$L_X L_Y - L_Y L_X = L_{[X,Y]}.$$

5. Define a 1-form on the punctured plane $\mathbb{R}^2 - \{0\}$ by

$$\omega = -\left(\frac{y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy.$$

Calculate $\int_C \omega$ for the circle of radius r centered at the origin.

6. a) Give the definition of a linear connection on a manifold M.

b) Is the Lie derivative a connection? Explain your answer.

7. a) Give the definitions of a $(0, s)$-tensor and a $(1, s)$-tensor on a differentiable manifold.

b) Let ∇ be a linear connection and Y a vector field. Show that $(\nabla Y)(X) = \nabla_X Y$ defines ∇Y as a $(1, 1)$-tensor.

8. a) State the definition of the curvature tensor R of a connection ∇.

b) Prove the first Bianchi identity for the curvature tensor:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$