Fall '06 Geometry Preliminary Exam

September 15, 2006

Answer **SIX** (6) of the following, including **at least two** (2) from among problems 6–10. You have three hours. Budget your time wisely.

1. (a) Define tensor, metric, and differential form.

   (b) Write down the (canonical) volume form on a Riemannian manifold on an open set U coordinatized by $\{x^i\}$.

2. Let G be a Lie group – a differentiable manifold with the structure of a group, such that the mapping $G \times G \to G$ defined by $(g, h) \mapsto gh^{-1}$ is differentiable.

   (a) Define the notion of invariant vector field, and give the construction which equates the tangent space at the identity with the space of invariant vector fields. (You may use left or right invariance.)

   (b) Use part (2a) to prove that the tangent bundle of a Lie group is trivial.

3. (a) Define *vector bundle*.

   (b) Prove that the tangent bundle is a vector bundle.

   (c) Define a connection on a vector bundle.

   (d) Let D and D' be two connections on the **tangent** bundle of a manifold M – so $D(X, Y) = D_X Y$, where X and Y are vector fields. Show that the *difference* of the two connections $D - D'$ is tensorial.

4. Let $M \subset \mathbf{R}^3$ be the subset of Euclidean three-space defined by the zeroes of the function $f(x, y) = x^2 + y^2 - z$.

   (a) Prove that M is a submanifold.

   (b) Define the induced metric on M and write it down explicitly in a coordinate system.

   (c) Calculate any nonzero component of the curvature tensor at the point $(0, 0, 0) \in M$.

5. Let $M \subset N$ be a Riemannian submanifold of codimension one (hypersurface) and fix a unit normal η on M.

   (a) Define the second fundamental form B on M (also known as H_η).

   (b) For X and Y tangent vector fields on M, show $\langle \overline{\nabla}_X \eta, Y \rangle = -B(X, Y)$
   Hint: Use the fact that $\langle Y, \eta \rangle = 0$.

6. (a) State the Hopf-Rinow theorem, using at least two equivalent hypotheses.
   (b) Using part (6a), prove that the torus
$$T^n = \{(z_1, ..., z_n) \in \mathbf{C}^n \ : \ |z_i| = 1 \text{ for all } i \ \} \subset \mathbf{C}^n$$
   is complete.

7. (a) Write down the geodesic equation.
   (b) Define the exponential map and prove that it is a local diffeomorphism in a neighborhood of the origin.

   Hint: At the zero vector, the derivative of the exponential map is the identity map (where we have identified the tangent space of a vector space with itself).

8. (a) Explain what a Jacobi field is and how one derives the equation which defines it. (You need not perform the derivation).
   (b) Explain what the length of a Jacobi field tells you about geodesics.
   (c) What does it mean if a nonzero Jacobi field vanishes at some point along a geodesic, and how does this help with classification theorems? Be as specific as possible.

9. Let $S^1 \subset \mathbf{R}^2$ be the unit circle with its induced metric, and denote by TS^1 the total space of its tangent bundle.
   (a) Choose coordinates for TS^1 and write down the geodesic vector field G on TS^1 in these coordinates.
   (b) Write down the equations of flow for an arbitrary vector field on a manifold, then pick a point in TS^1 and write down the equation for the (geodesic) flow defined by G starting at that point.
   (c) Solve the equation to find the path $\gamma : \mathbf{R}_{\geq 0} \to TS^1$. Compute $\pi \circ \gamma$, where $\pi : TS^1 \to S^1$ is the fiber map. Observe that $\pi \circ \gamma$ is a geodesic and explain how the point from part (9b) determines *which* geodesic.

10. Let $\gamma : [a, b] \to M$ be a smooth path on a Riemannian manifold, M.
   (a) Define the energy $E(\gamma)$.
   (b) Define what a variation of a path is.
   (c) Show that at the path γ, the variation of the energy in the direction of a vector field V along γ is
$$E'(0) = \int_a^b \langle V, \frac{D}{Dt}\frac{d\gamma}{dt} \rangle.$$

   Hint: You may assume the endpoints are fixed and that the variation is smooth – not just piecewise smooth – for all values of the parameter.

   Hint: You will use that the connection is symmetric and compatible with the metric.