Fall '07 Geometry Preliminary Exam
September 12, 2007

Answer **SIX** (6) of the following. You have three hours. Budget your time wisely.

1. (a) Define tensor, metric, and differential form.
   (b) Write down the (canonical) volume form on an oriented Riemannian manifold on an open set U coordinatized by $\{x^i\}$.
   (c) Define a *vector bundle*.
   (d) Define a connection on a vector bundle.
   (e) Let D and D' be two connections on a vector bundle V over a manifold M – so $D(X, s) = D_X s$, where X is a vector field on M and s is a section of V Show that the *difference* of the two connections $D - D'$ defines a vector bundle map $TM \otimes V \to V$.

2. Let G be a Lie group – a differentiable manifold with the structure of a group, such that the mapping $G \times G \to G$ defined by $(g, h) \mapsto gh^{-1}$ is differentiable.
   (a) Define the notion of invariant vector field, and give the construction which equates the tangent space at the identity with the space of invariant vector fields. (You may use left or right invariance.)
   (b) Use part (2a) to prove that the tangent bundle of a Lie group is trivial.
   (c) Find the invariant vector field of the (abelian) Lie group $(\mathbf{R}_{>0}, \times)$, and calculate its flow from an arbitrary point.

3. Let g be a metric on a Riemannian manifold M, and let $\pi : TM \to M$ be the tangent bundle. Let $U \subset M$ be an open set coordinaized by $\{x^i\}$ and let Γ^i_{jk} be the Christoffel symbols of the Levi-Civita connection in these coordinates. Recall that the connection defines a horizontal distribution on the total space of the tangent bundle. Give coordinates for TM on $\pi^{-1}(U)$ and find the horizontal lift of the vector field $\frac{\partial}{\partial x^i}$ to $\pi^{-1}(U)$ in your coordinates.

4. Let $M \subset \mathbf{R}^3$ be the subset of Eulidean three-space defined by the zeros of the function $f(x, y, z) = xy - z$.
   (a) Prove that M is a submanifold of $\mathbf{R}^3$.
   (b) Define the induced metric on M and write it down explicitly in a coordinate system.
   (c) Calculate the sectional curvature of M at $(0, 0, 0)$. (Recall that there is only one two-plane on a surface.)

5. The Hopf map $H : S^3 \to S^2$ can be written as follows, where $S^3 \subset \mathbf{R}^4 \cong \mathbf{C}^2$ is a sphere of radius 1 and $S^2 \subset \mathbf{R}^3 \cong \mathbf{R} \oplus \mathbf{C}$ is a sphere of radius 1/2. We write
$$H(z, w) = \left(\frac{1}{2}(|w|^2 - |z|^2), z\overline{w} \right).$$
   Prove that H is a Riemannian submersion, i.e. H_* is an isometry on $ker(H_*)^{\perp}$.

6. Let M be a Riemannian submanifold of N.

   (a) Define the second fundamental form $B : TM \times TM \to (TM)^{\perp}$.

   (b) Suppose that M is the fixed locus of an involution $\sigma : N \to N$ (i.e., $\sigma^2 = \mathrm{id}_N$) which is an isometry. Show that the M is a totally geodesic submanifold of N, i.e., the second fundamental form of M in N is zero.

7. (a) Let $f : M \to \mathbf{R}$ be a *proper*, distance-nonincreasing function on a Riemannian manifold, so $|f(x) - f(y)| \leq dist_M(x, y)$. Prove that M is complete. (Recall that "proper" means the preimage of a compact set is compact.)

   (b) Let M be a complete manifold. Define such a function, f.
   Hint: You may wind up using the triangle inequality.

8. Prove that a harmonic form on a compact manifold has minimum length (defined with the global inner product) in the cohomology class it represents.

9. Use the Poincaré lemma, and its version for ∂ and $\bar{\partial}$, to show that every point of a Kähler manifold X has a neighborhood U where the fundamental (Kähler) form Ω can be represented as

$$\Omega = \partial\bar{\partial}\Phi$$

   for a smooth, complex-valued function Φ on U. The function Φ is called a Kähler potential.

10. Let X be a complex manifold and $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. Let us denote $\mathcal{O}_X^*$ the subsheaf of holomorphic nowhere vanishing functions.

   (a) Show that the sequence of sheaves

$$0 \to \mathbf{Z} \to \mathcal{O}_X \xrightarrow{\mu} \mathcal{O}_X^* \to 1$$

   is exact.

   (b) Show that the squence of abelian groups

$$0 \to \mathbf{Z} \to \mathcal{O}_X(X) \xrightarrow{\mu} \mathcal{O}_X^*(X) \to 1$$

   is NOT exact for $X = \mathbf{C}^{\times}$ in the most right term (i.e. μ is not surjective).

   Here $\mathbf{Z}$ denotes both the group of integers and the constant sheaf, and $\mathbf{C}^{\times}$ denotes the complex plane with the origin removed.