Geometry Prelim Exam
Sept. 17, 1997
Work **six** *of the following problems*
Please list the six you want graded on the front of your blue book

1. Define *chart, atlas* and *manifold.* Define C^∞ *function* from one manifold to another and define *diffeomorphism.*

   If $f : M \to N$ is a C^∞ function from the manifold M to the manifold N prove that its graph $G = \{(x, f(x)) \in M \times N |\ x \in M\}$ has the structure of a manifold and that it is diffeomorphic to M.

2. Define *critical point* and *critical value* of a smooth function. State Sard's Theorem. Consider two C^∞ functions $f : \mathbb{R} \to \mathbb{R}$ and $g : S^1 \to S^1$. Is it possible that the set of critical values of f is dense in $\mathbb{R}$ or that the set of critical values of g is dense in S^1? In each case prove your answer; i.e. either prove it is not possible or construct an example.

   $f: \mathbb{R} \to \mathbb{R}$
   [illegible] that [illegible] critical [illegible] in $\mathbb{R}$

3. Define *transversal intersection* of two submanifolds.

   Suppose M^m and N^n are compact manifolds of dimension m and n respectively. Let $f : M \to N$ be a C^∞ function and suppose $y \in N$ is a regular value of f. Let $W = f^{-1}(y)$ so that W is a submanifold of M. Assume W is non-empty. If V is a submanifold of M which has a non-empty transverse intersection with W then give upper and lower bounds (in terms of m and n) on the possible dimension of V. Justify your answer. Give an example to show that at least one of these bounds is not valid if $W = f^{-1}(y)$ is a submanifold of M but $y \in N$ is not a regular value of f.

4. (a) State Stokes' Theorem for differential forms on a manifold.

   (b) Prove that Green's Theorem in the plane is a corollary of Stokes' theorem as you stated it. Green's theorem asserts that for a smooth simple closed curve C bounding a region D in the plane and for smooth functions $P(x, y), Q(x, y)$,

$$\int_C P dx + Q dy = \int\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy.$$

5. For a surface M in R^3 the *Levi-Civita* connection is defined by $D_X Y = \bar{D}_X Y + \langle L(X), Y\rangle$ where X and Y are tangent vector fields to M and $\bar{D}_X Y = (X(y_1), X(y_2), X(y_3))$ if $Y = (y_1, y_2, y_3)$. Prove that $D_X Y$ is also tangent to the surface M.

6. Let M be a smooth two dimensional submanifold of $\mathbb{R}^3$ with the induced Riemannian metric and let $\gamma(t),\ t \in [0, 1]$ be a smooth curve in M. Prove a tangent vector field Y is parallel along γ if and only if $\frac{dY}{dt}$ is normal to M.