Gauss-Bonnet Theorem

7. Let M be a smooth two dimensional Riemannian manifold and suppose the Gaussian curvature of M is strictly positive at every point. Consider a "geodesic triangle" in M, i.e. the image of an embedding of a Euclidean triangle with the property that the image of each side is geodesic segment. Prove that the sum of the interior angles at the corners of this triangle is strictly greater than π.

8. (a) State the first and second Cartan Structure equations.

(b) Let M be the hyperbolic plane, i.e. $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with Riemannian metric given by the tensor

$$\frac{dx \otimes dx + dy \otimes dy}{y^2}.$$

Prove that M has constant Gaussian curvature $K = -1$.

9. (a) State the definition of a Jacobi field along a geodesic in a Riemannian manifold.

Jacobi equation

(b) Show that if V is a Jacobi field along a geodesic C then its component tangent to C is linear, i.e., $\langle \dot{C}, V \rangle$ is a liner function along C.

10. (a) State the Hopf-Rinow Theorem.

(b)It is well known that a Riemannian manifold M is a metric space under the Riemannian distance function $d : M \times M \to [0, \infty)$. Show that M is complete if and only if every bounded set is precompact.

(c) Show that if M is complete, then it is compact if and only if the diameter of M is finite.