Preliminary Exam - Differential Geometry (D41/D43, Spring 99/00)

Do all problems.

Problem 1. (a) Define the universal cover of connected differentiable manifold. (b) Show that the universal cover of the unit circle $\left\{e^{i\theta} : 0 \le \theta < 2\pi\right\}$ is the real line $R = (-\infty, \infty)$.

Problem 2. (a) Define the orientability of a manifold. (b) Show that if a differentiable manifold M is covered by two charts U and V whose intersection $U \cap V$ is connected, then M is orientable.

Problem 3. (a) Define the tangent space $T_x M$ of a differentiable manifold M at a point $x \in M$. (b) Show that $T_x M$ has the same dimension as M.

Problem 4. (a) Define the torsion and curvature of a connection. (b) Define the Christoffel symbols Γ_{ij}^k of a connection. (c) Show that the connection is torsion-free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Problem 5. (a) Define the exterior derivative of a differential form, either invariantly or in local coordinates. (b) State Stoke's theorem.

Problem 6. Let M be a noncompact Riemannian manifold and $\{O_n\}$ a sequence of relative compact open subset of M such that $\overline{O}_n \subseteq O_{n+1}$ and $\cup_{n=1}^{\infty} O_n = M$. Define

$$d(x) = \lim_{n \to \infty} d(x, M \backslash O_n), \qquad x \in M,$$

where $d(x, A)$ is the Riemannian distance from x to set A. (a) Show that either $d(x) = \infty$ for all $x \in M$ or $d(x) < \infty$ for all $x \in M$. (b) if $d(x) = \infty$ for all $x \in M$, then M is complete.

Problem 7. State and prove Gauss' lemma about the geodesic polar coordinates of a Riemannian manifold.