PRELIMINARY EXAM IN GEOMETRY AND TOPOLOGY AUGUST 2021

INSTRUCTIONS: (1) There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then indicate which you would like graded; otherwise we will grade the first three you attempt in each section.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Differentiable Topology and the Fundamental Group

Do **three** of the following five problems.

(1) Classify all manifolds X for which:
- the universal cover of X is homeomorphic to S^4;
- the fundamental group of X is finite.

(2) Let M_k be an orientable surface of genus k.
 (a) Compute $[M_3, M_2]$, the set of homotopy-classes of maps from M_3 to M_2.
 (b) For each $g : M_3 \to M_2$, compute $\pi_n\big(\mathsf{Map}(M_3, M_2), g\big)$, the set of basepoint-preserving homotopy-classes of maps from $(S^n, *)$ to $(\mathsf{Map}(M_3, M_2), g)$.[1]

(3) Consider the polynomial
$$f = x^3 + xyz$$
in 3 real variables. Is the set of zeros $V(f) \subset \mathbb{R}^3$ a smooth manifold? Prove your answer.

(4) Prove that the inclusion $\mathsf{O}(n) \hookrightarrow \mathsf{GL}(n)$ is a homotopy equivalence.

(5) Consider the Stiefel manifold $\mathsf{V}_k(\mathbb{R}^n)$ of orthonormal k-frames $\{x_1, \dots, x_k\}$ in $\mathbb{R}^n$, topologized as a subspace of $(\mathbb{R}^n)^k$. Give a complete calculation of the Euler characteristic
$$\chi\big(\mathsf{V}_k(\mathbb{R}^n)\big)$$
for all n and k. Prove your answer.

[1]Here $\mathsf{Map}(M_3, M_2)$ is endowed with the compact-open topology.

Part II. Algebraic Topology

Do **three** of the following five problems.

(1) A space X is of *finite type* if $H_n(X, \mathbb{Z})$ is finitely generated as an abelian group for all n.

(a) Suppose there is a finite covering map $Y \to X$ and X is a finite simplicial complex. Show Y is of finite type.

(b) Give an example of finite type space X whose universal cover is not of finite type.

(2) Let M_n denote the orientable surface of genus n. Embed M_n into $\mathbb{R}^3$ in the usual way. Then M_n bounds a compact region X. Two copies of X, glued along their common boundary, form a 3-manifold N. Compute the integral homology of N.

(3) The universal coefficient theorem for homology gives a natural short exact sequence

$$0 \to H_n(X, \mathbb{Z}) \otimes A \to H_n(X, A) \to \mathrm{Tor}(H_{n-1}X, A) \to 0.$$

This is split for any abelian group A. Show by example that the splitting is not natural.

(4) Let $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$, $1 \leq n \leq \infty$ denote real and complex projective space respectively. There is a map $f : \mathbb{R}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ given by regarding both as quotients of $S^\infty = \cup S^n$, first by taking the quotient by the action of $\{\pm 1\}$ then by $S^1 \subseteq \mathbb{C}$.

(a) Show this map restricts to map $\mathbb{R}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^1$ which is surjective in integral cohomology.

(b) Conclude that f is surjective in integral cohomology in all degrees.

(5) Let M be a path-connected orientable closed manifold. Show that there is no decomposition up to homotopy equivalence

$$M \simeq X \vee Y$$

where $\vee$ denotes a one-point union and X and Y are two spaces with non-trivial reduced homology; that is,

$$\tilde{H}_*(X, \mathbb{Z}) \neq 0 \neq \tilde{H}_*(Y, \mathbb{Z}).$$

Part III. Differential Topology and Geometry

Do **three** of the following five problems.

(1) (a) State the Inverse Function Theorem.
(b) State Sard's theorem.
(c) State the Stokes theorem.

(2) Let G be a finite group acting smoothly on a smooth manifold M. Prove that there exists a Riemannian metric on M such that the action of G is by isometries.

(3) Do the following:
(a) Prove that every rank-n vector bundle E over $\mathbb{R}$ is trivializable. (I.e., there exists an isomorphism $E \cong \mathbb{R} \times \mathbb{R}^n$ of vector bundles over $\mathbb{R}$.)
(b) Classify rank-n vector bundles on S^1, for all n, up to isomorphism.
(c) Prove that the normal bundle of any knot $\varphi : S^1 \hookrightarrow S^3$ is trivializable.

(4) Consider the upper half plane $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$ with the hyperbolic metric

$$g_{11} = g_{22} = \frac{1}{x_2^2}$$
$$g_{12} = 0\,.$$

(a) Write the general Christoffel identies, an equation expressing the Christoffel symbols Γ_{ij}^k in terms of the metric g_{ij}.
(b) Calculate the Christoffel symbols Γ_{ij}^k for the associated Levi-Civita connection on $\mathbb{H}$.

(5) (a) Write the geodesic equation in terms of the Christoffel symbols, expressing the condition for a smooth parametrized curve $\gamma(t) = (u_1(t), \ldots, u_n(t)$ to be a geodesic.
(b) Consider the upper half plane $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$ with the hyperbolic metric $g_{11} = g_{22} = \frac{1}{x_2^2}$ and $g_{12} = 0$. Prove that for each constant r, the semi-circle of radius r parametrized as

$$x_1(t) = -r \cdot \mathsf{tanh}(t)$$
$$x_2(t) = r \cdot \mathsf{sech}(t)$$

is a geodesic in $\mathbb{H}$.[2]

[2] Recall $\mathsf{tanh}(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ and $\mathsf{sech}(t) = \frac{2}{e^t + e^{-t}}$.