PRELIMINARY EXAM IN GEOMETRY AND TOPOLOGY SPRING 2021

INSTRUCTIONS: (1) There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three, then indicate which you would like graded; otherwise we will grade the first three you attempt in each section.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Differentiable Topology and the Fundamental Group

Do **three** of the following five problems.

(1) For this question, you may make use of any standard results, so long as they are referenced explicitly.
  (a) Identify the Klein bottle K as quotient of a polygon.
  (b) Compute the fundamental group
$$\pi_1 K$$
  in terms of generators and relations.
  (c) Calculate the Euler characteristic $\chi(K)$.

(2) Let K again be the Klein bottle.
  (a) Construct a double covering map $\mathbb{T}^2 \to K$ from the torus to the Klein bottle.
  (b) Compute the induced map on fundamental groups
$$\pi_1 \mathbb{T}^2 \to \pi_1 K$$
  in terms of your presentation of $\pi_1 K$.

(3) Prove the following result:[1] If (X, d) is a complete metric space, every contraction $f : X \to X$ has a unique fixed point. (Recall that a mapping $f : X \to X$ is a contraction if there exists a real number $a < 1$ such that the inequality
$$d\big(f(x), f(y)\big) < a \cdot d(x, y)$$
holds for every pair of distinct points x and y in X.)

(4) Consider the unitary group $\mathsf{U}(n)$.
  (a) Prove that $\mathsf{U}(n)$ is a connected topological space.
  (b) Prove that $\mathsf{U}(2)$ is a product of spheres.
  (c) Calculate $\pi_1 \mathsf{U}(n)$ for all n.

(5) Consider $\mathsf{SO}(n)$ is the special orthogonal group.

[1] As you may recall, this is used in the proof the inverse function theorem.

(a) Prove that the universal cover of $\mathsf{SO}(4)$ is a product of spheres.
(b) Calculate the fundamental group $\pi_1\mathsf{SO}(4)$.
(c) Calculate the homology $\mathsf{H}_*(\mathsf{SO}(4), \mathbb{Z})$.

Part II. Algebraic Topology

Do **three** of the following five problems.

(1) Let $\phi : C_\bullet X \to C_\bullet X$ be a *natural* chain map on the chain complex of singular simplices of X. Suppose $\phi : C_0 X \to C_0 X$ is the identity. Show ϕ is chain homotopic to the identity.

(2) Let $f : X \to X$ be a self-map and define the mapping torus T_f of f to the quotient space of $X \times [0,1]$ obtained by identifying $(x, 0)$ with $(f(x), 1)$. Show there is a long exact sequence in homology

$$\longrightarrow H_n X \oplus H_n X \xrightarrow{\Phi} H_n X \oplus H_n X \longrightarrow H_n T_f \longrightarrow H_n X \oplus H_n X \longrightarrow$$

where $\Phi(a, b) = (a + b, a + f_* b)$. Use this to calculate $H_* T_f$ where $f : S^1 \to S^1$ is a degree n map.

(3) Let $f : X \to Y$ be a map of finite connected CW complexes. Here "finite" means that it has finitely many cells. Prove that if $f_* : H_*(X, \mathbb{Z}/p) \to H_*(Y, \mathbb{Z}/p)$ is an isomorphism for all primes p, then $f_* : H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$ is an isomorphism.

(4) Prove $\mathbb{C}\mathrm{P}^n$ cannot be a topological group if $n < \infty$. To do this, notice that if it were, there would be a multiplication $m : \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n$ and an identity element $e \in \mathbb{C}\mathrm{P}^n$. The map

$$\{e\} \times \mathbb{C}\mathrm{P}^n \xrightarrow{\subseteq} \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n \xrightarrow{m} \mathbb{C}\mathrm{P}^n$$

would be the identity, and similarly for the inclusion of $\mathbb{C}\mathrm{P}^n \times \{e\}$. Now make some cohomology calculations.

(5) Let T_n denote the n-holed torus. Show there is a map $f : T_n \to T_m$ with the property that $f : H_2 T_n \to H_2 T_m$ is an isomorphism if and only if $n \geq m$.

Part III. Differential Topology and Geometry

Do **three** of the following five problems.

(1) (a) State the Inverse Function Theorem.
(b) State Sard's theorem.
(c) State the Stokes theorem.

(2) Consider the Stiefel manifold $\mathsf{V}_k(\mathbb{R}^n)$ of orthonormal k-frames $\{x_1, \ldots, x_k\}$ in $\mathbb{R}^n$, topologized as a subspace of $(\mathbb{R}^n)^k$. Give a complete calculation of the Euler characteristic

$$\chi\big(\mathsf{V}_k(\mathbb{R}^n)\big)$$

for all n and k. Prove your answer.

(3) Determine the set of isomorphism classes of 3-dimensional real vector bundles on $\mathbb{R}\mathrm{P}^3$.

(4) Consider the upper half plane $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$ with the hyperbolic metric

$$g_{11} = g_{22} = \frac{1}{x_2^2}$$
$$g_{12} = 0\,.$$

(a) Write the general Christoffel identies, an equation expressing the Christoffel symbols Γ_{ij}^k in terms of the metric g_{ij}.
(b) Calculate the Christoffel symbols Γ_{ij}^k for the associated Levi-Civita connection on $\mathbb{H}$.

(5) (a) Write the geodesic equation in terms of the Christoffel symbols, expressing the condition for a smooth parametrized curve $\gamma(t) = (u_1(t), \ldots, u_n(t)$ to be a geodesic.
(b) Consider the upper half plane $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$ with the hyperbolic metric $g_{11} = g_{22} = \frac{1}{x_2^2}$ and $g_{12} = 0$. Prove that for each constant r, the semi-circle of radius r parametrized as

$$x_1(t) = -r \cdot \mathsf{tanh}(t)$$
$$x_2(t) = r \cdot \mathsf{sech}(t)$$

is a geodesic in $\mathbb{H}$.[2]

[2]Recall $\mathsf{tanh}(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ and $\mathsf{sech}(t) = \frac{2}{e^t + e^{-t}}$.