Preliminary Exam in Topology & Geometry Spring 2022

Instructions:

(1) There are **three** parts to this exam: I Analysis on Manifolds, II Introduction to Algebraic Topology, and III Cohomology. Do **three** problems from each part. If you attempt more than three problems in each section, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Analysis on Manifolds

Do **three** of the following five problems.

Problem 1. (a) State the following theorems: (1) the inverse function theorem; (2) Sard's theorem; (3) the Stokes theorem. (b) Sketch a direct proof of Sard's theorem for a smooth function $f : \mathbb{R}^1 \to \mathbb{R}^1$.

Problem 2. The Poincaré upper half plane is the half space $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ equipped with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

(a) Calculate the Christoffel symbols of the Levi-Civitá connection of $\mathbb{H}$. (b) Find the geodesic passing through the point $(0, 1)$ with its tangent line at this point parallel to the x-axis.

Problem 3. (a) Define an orientable (smooth) manifold. (b) Define the real projective space $\mathbb{RP}^n$. (c) Show that a real projective space of odd dimension is orientable.

Problem 4. Let M be a smooth manifold equipped with a connection $\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$, where $\Gamma(M)$ is the space of vector fields on M. (a) Describe how to extend the connection from vector fields to tensor fields $\nabla : \Gamma(M) \times \Gamma^{p,q}(M) \to \Gamma^{p,q}(M)$. (b) Suppose that θ is a 2-form on M. Show that

$$(\nabla_X \theta) = X\theta(Y, Z) - \theta(\nabla_X Y, Z) - \theta(Y, \nabla_X Z).$$

(c) Suppose that the connection is torsion free and that η is a 1-form on M . Show that

$$(d\theta)(X, Y) = (\nabla_X \theta)(Y) - (\nabla_Y \theta)(X).$$

Problem 5. (a) Define a submersion of a smooth manifold into another smooth manifold. (b) Suppose that M is a smooth compact manifold. Show that there does not exist a submersion from M to $\mathbb{R}^n$ for any $n \geq 1$.

Part II. Introduction to Algebraic Topology.

Do **three** of the following five problems.

Problem 1. (a) Define pointed homotopy equivalence for continuous maps $f\colon X \to Y$ with basepoints $x \in X$ and $y \in Y$ and show that it is an equivalence relation. (b) Sketch a proof that the set of pointed homotopy equivalence classes of maps from S^n to X is a group for $n \geqslant 1$ and that it is abelian for $n \geqslant 2$ (for any chosen basepoints of S^n and X).

Problem 2. Prove the Brouwer fixed point theorem: every continuous map from D^n to itself has a fixed point for $n \geqslant 0$.

Problem 3. Prove that for every finite group G there is a path-connected manifold M with $\pi_1(M) = G$.

Problem 4. Let T be the torus $S^1 \times S^1$ and let X be T minus a point. (a) Compute $\pi_1(X)$. (b) Compute $\mathrm{H}_*(T, \mathbf{Z})$ and $\mathrm{H}_*(X, \mathbf{Z})$.

Problem 5. (a) Define the CW chain complex $\mathrm{C}_\bullet^{\mathrm{CW}}(X; \mathbf{Z})$ for CW complexes X. (b) Prove that the homology groups $\mathrm{H}_*^{\mathrm{CW}}(X; \mathbf{Z})$ are naturally isomorphic to the singular homology groups $\mathrm{H}_*(X; \mathbf{Z})$ of X.

Part III. Cohomology.

Do **three** of the following five problems.

Problem 1. Prove the Poincaré lemma: if $p > 0$, every differentiable p-form ω on $\mathbf{R}^n$ is exact: $\omega = \mathrm{d}\tau$ for some τ.

Problem 2. Let Γ_g be the oriented closed surface of genus g. Compute (with justification for the product structure) the singular cohomology ring $\mathrm{H}^*(\Gamma_g, \mathbf{Z})$.

Problem 3. Let X be the space $\{0, 1, 2\}$ with the topology $\{\varnothing, \{0\}, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}\}$. (a) Compute the Godement resolution of the constant sheaf $\underline{\mathbf{Z}}$ on X. (b) Compute the sheaf cohomology groups $\mathrm{H}^*(X, \underline{\mathbf{Z}})$.

Problem 4. Compute the singular cohomology ring $\mathrm{H}^*(T \vee T; \mathbf{Z})$ of $T \vee T$, where $T = S^1 \times S^1$ is the torus.

Problem 5. Let M be a path-connected topological manifold with the property that every element of $\pi_1(M)$ has finite *odd* order. Prove that M is orientable.