Geometry and Topology Prelim
June 2013

Do at least two problems from each group, and more if you can.

Group I

1. Let (X, x_0) be a path-connected space with a basepoint and let $\pi = \pi_1(X, x_0)$.
   (a) Suppose $q : Y \to X$ a covering map. Show the set $q^{-1}(x_0)$ has a left action by π.
   (b) Now suppose X has a universal cover and A is a left π-set. Find a covering map $q : Y \to X$ with $q^{-1}(x_0) \cong A$ as π-sets.
2. Let T be the torus and let T_0 be T with a small open disk deleted. Let M be the Möbius band. Define $X = T_0 \cup_{S^1} M$, where we identify the boundary circles of T_0 with the boundary circle of M. Then X is the connect sum of T and $\mathbb{R}\mathrm{P}^2$. Compute $\pi_1(X)$ and $H_*(X, \mathbb{Z})$.
3. Let $X \subseteq \mathbb{R}^3$ be the union of the unit sphere S^2 and the unit disk in the xy-plane. Give X the structure of a CW complex and write down the resulting CW chain complex for computing the homology of X.

Group II

1. Consider the two vector fields on $\mathbb{R}^2$ given by
$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y}.$$
   (a) Find the smooth $\mathbb{R}$-action on $\mathbb{R}^2$ whose infinitesimal generator is X. Namely, find smooth $\theta_t : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $\theta_0 = \mathrm{Id}$ and
$$\left.\frac{d}{dt}\right|_{t=0} \theta_t(p) = X_p, \quad \text{for all } p \in \mathbb{R}^2.$$
   (b) Find $\mathcal{L}_X Y$ using (a) and the definition of the Lie derivative
$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} (\theta_{-t})_* Y.$$
   (c) Compute $[X, Y]$ directly and check that your answer is the same as your answer to (b).

2. Let (M, g) be a Riemannian manifold of dimension n.

   (a) Fix a point p in M. Explain why we can choose a coordinate system $(x^1, \ldots, x^n)$ centered at p with the property that the components of the tensor g at the point p are given by
   $$g_{ij} = \delta_{ij}$$
   where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.
   (You may want to use the following fact from linear algebra: if A is a positive definite symmetric matrix then there exists an invertible $n \times n$ matrix B such that $B^T AB = I$, where I is the identity matrix.)

   (b) Let $(x^1, \ldots, x^n)$ be the coordinates from (a). Define a new coordinate system $(\tilde{x}^1, \ldots, \tilde{x}^n)$ in a neighborhood of the point p by $x^k = \tilde{x}^k - \frac{1}{2}\sum_{i,j} \Gamma^k_{ij}(p)\tilde{x}^i\tilde{x}^j$, where $\Gamma^k_{ij}(p)$ are the Christoffel symbols of the Levi-Civita connection of g at the point p in the $x^1, \ldots, x^n$ coordinates. Show that for all i, j, k,
   $$\frac{\partial}{\partial \tilde{x}^k}\tilde{g}_{ij} = 0 \quad \text{at } p,$$
   where $\tilde{g}_{ij}$ are the components of the metric g in the $\tilde{x}^1, \ldots, \tilde{x}^n$ coordinates.
   (Recall the formula: $\Gamma^k_{ij} = \frac{1}{2}\sum_q g^{kq}(\partial_i g_{jq} + \partial_j g_{iq} - \partial_q g_{ij})$.)

3. Let M be a smooth manifold of dimension $2n$. We say that a 2-form ω on M is *symplectic* if $d\omega = 0$ and ω^n is a nowhere vanishing $2n$-form on M.

   (a) Show that if M is compact with no boundary then no symplectic form ω on M is exact.

   (b) Show explicitly that $\omega = \sum_{i=1}^n dx^i \wedge dx^{n+i}$ is an exact symplectic form on $M = \mathbb{R}^{2n}$. (Recall that a form α is *exact* if $\alpha = d\beta$ for some form β.)

Group III

1. Show that $\mathbb{C}\mathrm{P}^2 \times S^4$ and $\mathbb{C}\mathrm{P}^4 \vee S^4$ are not homotopy equivalent.

2. Let M and N be two compact orientable n-manifolds without boundary and suppose there is a map $f : M \to N$ so that $f_* : H_n(M, \mathbb{Z}) \to H_n(N, \mathbb{Z})$ is an isomorphism. Show $f^* : H^*(N, \mathbb{Q}) \to H^*(M, \mathbb{Q})$ is one-to-one.

3. Let γ_n be the tautological bundle over $\mathrm{Gr}_n(\mathbb{C}^\infty)$. Then a typical element in $E(\gamma_n)_0$ is a pair (W, v) where $W \subseteq \mathbb{C}^\infty$ is a complex n-dimensional subspace and $0 \neq v \in W$. Define a map
   $$q : E(\gamma_n)_0 \to \mathrm{Gr}_{n-1}(\mathbb{C}^\infty)$$
   sending (W, v) to W_v where W_v is perpendicular to v. Assume you know that q is a homotopy equivalence. Then use induction and the Gysin sequence to prove that there are elements $c_i \in H^{2i}(\mathrm{Gr}_n(\mathbb{C}^\infty), \mathbb{Z})$, $i \leq i \leq n$, and an isomorphism
   $$\mathbb{Z}[c_1, \cdots, c_n] \cong H^*(\mathrm{Gr}_n(\mathbb{R}^\infty), \mathbb{Z}).$$