Geometry and Topology Preliminary Examination
Northwestern University
Spring 2014

Do at least two problems from each group, and more if you can.

Group I

1) Give an example of a covering space $X \to Y$ where Y is the wedge of three circles and $\pi_1(X)$ is the dihedral group $a^2 = 1, b^4 = 1$, $aba = b^3$.

2) Fix a natural number $N > 0$. Let

$$X = \{(u, \zeta) \mid u \in U(2, \mathbb{C}), \zeta \in \mathbb{C}, \det(u) = \zeta^N\}.$$

Let p be the projection $(u, \zeta) \mapsto u$. Show that there is no continuous map $q : U(2, \mathbb{C}) \to X$ such that $pq = \mathrm{id}$.

3) Let $D = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ and let

$$X = (D \times S^1) - (L_1 \cup L_2 \cup L_3 \cup L_4),$$

where

$$L_1 = \left\{(z_1, z_2) \;\middle|\; z_1 = -\frac{1}{2}\right\}, \quad L_2 = \left\{(z_1, z_2) \;\middle|\; z_1 = \frac{1}{2}\right\},$$

$$L_3 = \left\{(z_1, z_2) \;\middle|\; \left|z_1 - \frac{1}{2}\right| = \frac{1}{2},\ z_2 = 1\right\}, \quad L_4 = \left\{(z_1, z_2) \;\middle|\; \left|z_1 + \frac{1}{2}\right| = \frac{1}{2},\ z_2 = 1\right\}.$$

Compute $\pi_1(X)$.

Group II

1) Let G be a (finite-dimensional, not necessarily connected) Lie group, with identity element $e \in G$. Let $m : G \times G \to G$ be the group multiplication map.

(a) Via the usual identification $T_{(e,e)}(G \times G) \cong T_eG \oplus T_eG$, show that $dm_e : T_eG \oplus T_eG \to T_eG$ is given by

$$dm_e(X, Y) = X + Y,$$

for every $X, Y \in T_eG$.

Let now $i : G \to G$ be the inversion map of G.

(b) Show that for every $X \in T_eG$ we have

$$di_e(X) = -X.$$

2) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth positive function, and consider the surface of revolution

$$M = \{(f(u) \cos v, f(u) \sin v, u) \in \mathbb{R}^3 \mid u \in \mathbb{R}, 0 \leqslant v < 2\pi\}.$$

(a) Show that M is a submanifold of $\mathbb{R}^3$.

(b) Let $\iota : M \hookrightarrow \mathbb{R}^3$ be the inclusion. Using (u, v) as global coordinates on M, write down the metric $g = \iota^* g_{\text{Eucl}}$ induced from the Euclidean metric on $\mathbb{R}^3$.

(c) Write down the same metric explicitly when $f(x) = e^x$.

3) Consider the vector fields

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

in $\mathbb{R}^3$ with the standard coordinates (x, y, z).

(a) Find local coordinates (u, v, w) in a neighborhood of $(x, y, z) = (1, 0, 0)$, such that in these coordinates we have

$$X = \frac{\partial}{\partial u}, \quad Y = \frac{\partial}{\partial v}.$$

(b) Is it possible to do the same for the vector fields

$$X' = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad Y' = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}?$$

Group III

1) Compute $H^k(\mathbb{RP}^8; \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$, for all k.

2) Show that if $\pi : \mathbb{CP}^{2n} \to X$ is a covering space, then $X = \mathbb{CP}^{2n}$ and π is the identity.

3) Let M be a compact orientable manifold of dimension $n \geqslant 2$, and $p \in M$. Suppose you know the de Rham cohomology groups of M, determine those of $M \backslash \{p\}$.