Preliminary Examination in Geometry/Topology
September 14, 2009

1. Geometry

Answer 3 of the following questions.

(1) (a) Define the *torsion* of a connection on the tangent bundle of a manifold M.
(b) Prove that there exists a unique torsion-free connection on the tangent bundle of a Riemannian manifold M which is compatible with the metric.
(c) If ∂_i, ∂_j is a coordinate frame of vector fields, give a formula for $\nabla_{\partial_i} \partial_j$ in terms of the components $g_{ij} = g(\partial_i, \partial_j)$ of the metric.

(2) Let G be a compact Lie group.
(a) Show that G has a bi-invariant Riemannian metric σ.
(b) Show that the integral curves of left-invariant vector fields on G are geodesics for σ.
(c) Show that if Z is a left-invariant vector field on G, then $\nabla_Z Z = 0$, where ∇ is the Levi-Civita connection for σ.
(d) Show that if X and Y are left-invariant vector fields on G,

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

(3) (a) State Cartan's formula for the Lie derivative of a differential form.
(b) Let Y be an integrable vector field on M, and let $\phi_t : M \to M$ be the corresponding one-parameter family of diffeomorphisms. How does the operator $\phi_t^* : \Omega^k(M) \to \Omega^k(M)$ relate to the Lie derivative by Y?
(c) Suppose that ω is a closed two-form on a manifold M and $f : M \to \mathbb{R}$ is a differentiable function. If there is an integrable vector field Y_f satisfying $df(X) = \omega(Y_f, X)$ for all vector fields X, show that its flow preserves ω.

(4) Consider the distribution $\mathcal{D}$ defined by the following two vector fields on $\mathbb{R}^3$ with coordinates x, y, z:

$$V_1 = \partial_y + z\partial_x$$
$$V_2 = \partial_z + y\partial_x$$

(a) Express the distribution $\mathcal{D}$ as the kernel of a closed one-form.
(b) Show that the distribution is integrable.
(c) Conclude (how?) that we can find an integral manifold M_p through every point $p \in \mathbb{R}^3$, and find M_p for $p = (1, 0, 0)$.

2. Topology

Answer 3 of the following questions.

(5) Let X be a topological space which can be written as

$$X = U_1 \cup U_2 \cup U_3$$

with each U_i open in X. Suppose U_i and $U_i \cap U_j$ are contractible for $1 \le i, j \le 3$. Show

$$\tilde{H}_n X \cong \tilde{H}_{n-2}(U_1 \cap U_2 \cap U_3).$$

(6) Let $p : \tilde{X} \to X$ be the universal cover of a path-connected and locally-path connected space X and let $A \subseteq X$ be a path-connected and locally path-connected subspace. Let $\tilde{A}$ be a path component of $p^{-1}(A)$. Show that $\tilde{A} \to A$ is a covering space and that the image of

$$\pi_1(\tilde{A}, b) \to \pi_1(A, a)$$

is the kernel of $i_* : \pi_1(A, a) \to \pi_1(X, a)$. Here $b \in \tilde{A}$ is any basepoint and $a = p(b)$.

(7) Let X be the topological space obtained as the quotient of the sphere S^2 under the equivalence relation $x \sim -x$ for x in the equatorial circle.

(a) Describe a CW complex whose underlying space is X.
(b) Write down the CW chain complex of X.

(8) Let X be the space obtained from a torus $T = S^1 \times S^1$ be attaching a Möbius band M by a homeomorphism from the boundary circle of M to $S^1 \times \{x_0\} \subseteq T$. Compute $\pi_1(X, x_0)$.