GEOMETRY AND TOPOLOGY PRELIMINARY EXAM, JUNE 2010.

ANSWER 6 QUESTIONS, INCLUDING AT LEAST ONE OF 4 AND 5.

Question 1.
Let G be a discrete group, and X a connected space.

(1) Assuming any path and homotopy lifting properties you need, explain how to construct a group homomorphism $\pi_1(X,x) \to G$ from a principal G bundle on X.
(2) The fundamental group of $S^1 \vee S^1$ is the free group on two generators γ_1 and γ_2. Construct (explicitly) a principal $\mathbb{Z} \times \mathbb{Z}$ bundle on $S^1 \vee S^1$ such that the associated group homomorphism $\pi_1(S^1 \vee S^1) \to \mathbb{Z} \times \mathbb{Z}$ sends

$$\gamma_1 \to (1,0)$$
$$\gamma_2 \to (0,1).$$

Question 2.
Let $a, b \in \mathbb{RP}^2$ be two distinct points.

Let X be the space quotient of $\mathbb{RP}^2 \times \{1,2,3\}$ by the relations $(b,1) \sim (a,2)$, $(b,2) \sim (a,3)$, $(b,3) \sim (a,1)$.

Calculate the fundamental group of X, and hence classify all 3-fold connected covers of X.

Question 3. (1) Using the coordinate definition of the exterior derivative, prove the formula

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),$$

where X and Y are vector fields, and ω is a 1-form on a manifold, M.
(2) Suppose $M = G = GL(2,\mathbb{R})$. Define left-invariant vector fields X, Y on M, and a left-invariant 1-form ω on M, by the formulae

$$X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \omega_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + b - d.$$

Here $1 \in G$ is the identity, and we have identified the tangent space to G at the identity with the Lie algebra of G, i.e. the set of 2×2 matrices. Calculate $d\omega(X,Y)$ as a function on G.

Question 4.
Consider the distribution on $M = \left\{(x,y,z) \in \mathbb{R}^3 \middle| x > 0, y > 0\right\}$ given by

$$\Delta_{(x,y,z)} = \text{Span}\left\{ y\frac{\partial}{\partial x} + xy\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z} \right\}.$$

(1) Show that this distribution is integrable.
(2) Describe the maximal integral submanifolds.

Question 5.
Let M be a compact Riemannian manifold.

(1) What does it mean for a smooth map $f : (0, t) \to M$ to be a geodesic?
(2) Suppose that M is two-dimensional. Let $\sigma : M \to M$ be an isometry which satisfies $\sigma^2 = 1$. Suppose that the fixed point set $\gamma = \{x \in M \mid \sigma(x) = x\}$ is a connected one-dimensional submanifold of M.
Show that γ is the image of a geodesic.
(3) Let
$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1, \text{ and } z > 0\}.$$
Show that, for every straight line through the origin $L \subset \mathbb{R}^2$, the set
$$\{(x, y, z) \in M \mid (x, y) \in L\}$$
is a geodesic in M.

Question 6.
Let G be a finite group acting freely on a manifold M (this means that a non-identity element of G has no fixed points).

(1) Prove that M/G is a manifold.
(2) Prove that
$$H^i_{dR}(M/G) = H^i_{dR}(M)^G$$
where $H^i_{dR}(M)^G$ is the fixed points of the G action on $H^i_{dR}(M)$.
(3) Use this result to show that, if N is a compact, connected n dimensional manifold which is non-orientable,
$$H^n_{dR}(N) = 0.$$

Question 7.
If M, N are connected oriented manifolds of the same dimension. Let M' (respectively, N') be the manifold with boundary obtained by removing a small open ball from M (respectively, N). Let $M\#N$ be the manifold obtained by gluing the boundary sphere of M' to that of N', using an orientation reversing diffeomorphism.

Calculate the de Rham cohomology ring of $\left(S^1 \times S^3\right) \#\mathbb{CP}^2$.

Question 8.
Let Σ_g denote the compact oriented surface of genus g. Let
$$X = \Sigma_g \setminus \{p_1, \ldots, p_k\}$$
where the p_i are distinct points in Σ_g.

(1) Calculate the compactly supported de Rham cohomology of X.
(2) Is it true that every class in $H^i_c(X)$ can be represented as the fundamental class of some submanifold?
(3) Using intersection theory, or otherwise, calculate the ring structure on $H^*_c(X)$.