GEOMETRY AND TOPOLOGY PRELIMINARY EXAM, SEPTEMBER 2010.
ANSWER 9 QUESTIONS.

Question 1.
Let X, Y be connected locally contractible topological spaces, and suppose that

(1) The universal cover of Y is contractible.
(2) The fundamental group of X is trivial.

Show that every continuous map $f : X \to Y$ is homotopic to a constant map.

Question 2.
Let $Y \to X$ be a covering space of a topological space X. Let $y \in Y$ and let $x = p(y)$.

(1) Outline a construction of an action of the group $\pi_1(X, x)$ on the set $p^{-1}(x)$ (you may assume any path-lifting properties you need, as long as they are stated clearly).
(2) How do $\pi_1(Y, y)$, $\pi_0(Y)$, $\pi_1(X, x)$ and $p^{-1}(x)$ relate?

Now, let S be any set with an action of the group $\pi_1(X, x)$. Construct a covering space Y_S of X with a $\pi_1(X, x)$-equivariant isomorphism

$$S \cong p^{-1}(x).$$

(You may assume the existence of a universal cover, if necessary).

Question 3.
Let S_k be the space obtained from the sphere S^2 by

(1) Removing k disjoint open discs from S^2, to leave a manifold whose boundary is k circles;
(2) Gluing a Möbius band onto each circle (which we can do, as the boundary of a Möbius band is also a circle).

Use Van Kampen's theorem to calculate $\pi_1(S_k)$ for each $k > 0$.

Question 4. (1) State van Kampen's theorem, allowing you to compute the fundamental group of a space X written as a union of two open subsets U, V whose intersection is connected.
(2) Let

$$G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$$

be a finitely presented group, with generators g_i and relations r_j. Construct a space X with $\pi_1(X) = G$. (You should prove that $\pi_1(X)$ has this property).

Question 5.
Let

$$f : \mathrm{Mat}(n, n) \times \mathrm{Mat}(n, n) \to \mathrm{Mat}(n, n)$$

be the map of multiplication, $f(A, B) = A \cdot B$, where A and B are $n \times n$ matrices. Since $\mathrm{Mat}(n, n)$ is a vector spaces, its tangent bundle is trivial

$$T\,\mathrm{Mat}(n, n) \cong \mathrm{Mat}(n, n) \times \mathrm{Mat}(n, n) = \{(A; X), A, X \in \mathrm{Mat}(n, n)\}.$$

(1) Describe the derivative $Df(A, B; X, Y)$ of f at the point $(A, B; X, Y) \in T(\mathrm{Mat}(n, n) \times \mathrm{Mat}(n, n))$.
(2) Let g be the Riemannian metric

$$g_A(X, Y) = tr(X \cdot Y).$$

Compute the pullback $f^*(g)$ explicitly.

Question 6.
Let M be the sphere $x^2 + y^2 + z^2 = 1$ with spherical coordinates

$$x = \cos(\theta)\sin(\phi), \quad y = \sin(\theta)\sin(\phi), \quad z = \cos(\phi),$$

for $\theta \in [0, 2\pi)$, $\phi \in [0, \pi]$. Let dA denote the usual area form

$$dA = \sin(\theta) d\theta d\phi.$$

Using a coordinate system for the northern and southern hemisphere, calculate

$$\int_M e^{af(x,y,z)} dA$$

where a is a number, and $f(x, y, z) = z$ is the height function on the sphere.

Question 7.
Consider the distribution on $\mathbb{R}^3$ given by

$$\Delta_{(x,y,z)} = \mathrm{Span}\left\{ y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right\}.$$

(1) Show that this distribution is integrable.
(2) Describe the maximal integral submanifolds.

Question 8.
Let M be a Riemannian manifold.

(1) What does it mean for a connection on M to be *symmetric*?
(2) What does it mean for a connection on M to be *compatible with the metric*?
(3) Suppose Γ is the connection on $\mathbb{R}^n$ whose covariant derivative is given by

$$D_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k, \quad \partial_i = \frac{\partial}{\partial x_i},$$

and g is a Riemannian metric on $\mathbb{R}^n$ given by

$$g(\partial_i, \partial_j) = g_{ij}.$$

Derive the formula for the coordinates Γ_{ij}^k in terms of the metric g_{ij}, where Γ is the unique symmetric connection compatible with the metric.

Question 9.
Let X be a vector field on a manifold M, induced by $g_t : M \to M$ so that

$$(X \cdot f)(x) = \left.\frac{d}{dt}\right|_{t=0} f(g_t(x)).$$

(1) If Y is another vector field, the *Lie Derivative* $\mathcal{L}_X Y$ is defined by

$$(\mathcal{L}_X Y)_p = \left.\frac{d}{dt}\right|_{t=0} g_{-t*}(Y),$$

where $g_{-t*}(Y)$ is the pushforward of Y through g_{-t}. Prove that $(\mathcal{L}_X Y)(f) = X \cdot (Y \cdot f) - Y \cdot (X \cdot f)$.

(2) If ω is a differential form, we define

$$(\mathcal{L}_X \omega)_p = \left.\frac{d}{dt}\right|_{t=0} (g_t^* \omega)_p\,,$$

where $g_t^*(\omega)$ is the pullback of ω. Prove that

$$d\mathcal{L}_X \omega = \mathcal{L} d\omega$$

for ω a differential form on $\mathbb{R}^n$.

Question 10.
Let U, V be open subsets of M which cover M. Write out the Mayer-Vietoris sequence for the cohomology of M.

Let $T = S^1 \times S^1$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

be a 2×2 integer matrix with determinant one. Let

$$\phi_A : T \to T$$

be the map defined by

$$\phi_A\,(\theta, \sigma) = (a\theta + b\sigma, c\theta + d\sigma)\,,$$

where $\theta, \sigma \in \mathbb{R}/\mathbb{Z}$ are the angle coordinates $T = S^1 \times S^1$.

Let

$$M = T \times [0,1]/\sim$$

where $\sim$ is the equivalence relation identifying $(t, 0)$ with $(\phi_A(t), 1)$.

Calculate the de Rham cohomology of M.

Question 11. (1) State Stoke's theorem.

(2) Let M be a manifold, and $\omega \in \Omega^r(M)$ be an r-form. Suppose that

$$\int_\Sigma \omega = 0$$

for all submanifolds r of Σ which are diffeomorphic to a sphere.

Show that $\mathrm{d}\omega = 0$.

Question 12. (1) Let $\phi, \psi : C^* \to D^*$ be cochain maps between two cochain complexes. What does it mean for ϕ, ψ to be cochain homotopic?

(2) Let M, N be smooth manifolds, and let $f, g : M \to N$ be smooth maps. Let $F : M \times [0,1] \to N$. Use F to construct a cochain homotopy between the two cochain maps

$$f^*, g^* : \Omega^*(N) \to \Omega^*(M).$$

Question 13. (1) Use the Mayer-Vietoris sequence to compute the de Rham cohomology of $\mathbb{CP}^n$.

(2) Using intersection theory, or otherwise, calculate the ring structure on $H^*_{dR}(\mathbb{CP}^n)$.