GEOMETRY/TOPOLOGY PRELIMINARY EXAMINATION, JUNE 2026

INSTRUCTIONS:

- There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.
- In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I

Do **three** of the following five problems.

(1) Let M be a smooth compact manifold, and denote by $\mathfrak{X}(M)$ the space of smooth vector fields on M.

 (a) For a vector field $X \in \mathfrak{X}(M)$ provide the definition of the flow ϕ_t^X generated by X.

 (b) Let $S \subset M$ be an embedded submanifold, and suppose that $X, Y \in \mathfrak{X}(M)$ are tangent to S at every point of S. Show that $[X, Y]$ is tangent to S.

(2) Let $M, N \subset \mathbb{R}^k$ be compact embedded submanifolds of dimensions m, n respectively, with $m + n < k$. Prove that there exists a set $Z \subset \mathbb{R}^k$ of measure zero such that if $v \in \mathbb{R}^k \setminus Z$, then the translation $M + v$ is disjoint from N. You may use Sard's theorem.

(3) Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ be the upper half space. Define the vector fields

$$X = \frac{\partial}{\partial x}, \qquad Y = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

 (a) Compute the Lie derivative $L_X Y$, and determine whether the distribution $\mathcal{D}$ spanned by X, Y is involutive.

 (b) Do there exist local coordinates u, v, w on a coordinate chart containing the point $(0, 0, 1)$ on H such that $X = \frac{\partial}{\partial u}$ and $Y = \frac{\partial}{\partial v}$?

(4) Let M be a smooth manifold, and ∇, ∇' be two connections on TM. Define $A(X, Y) = \nabla_X Y - \nabla'_X Y$.

 (a) Prove that A is a $(1, 2)$-tensor.

 (b) Suppose that ∇, ∇' are torsion free, and in addition $\nabla_X X = \nabla'_X X$ for all vector fields X. Show that $\nabla = \nabla'$.

(5) Let (M, g) be a Riemannian manifold, and let $\gamma : \mathbb{R} \to M$ be a geodesic.

 (a) Let Γ^i_{jk} denote the Christoffel symbols of g in local coordinates $x^1, \ldots, x^n$. Write the geodesic equation for γ in local coordinates in terms of the Christoffel symbols.

 (b) Suppose that $M = \mathbb{R} \times S^1$, with local coordinates x, θ, and metric given by

$$g = dx \otimes dx + h(x)^2 d\theta \otimes d\theta.$$

 The only nonzero Christoffel symbols are $\Gamma^x_{\theta\theta}, \Gamma^\theta_{\theta x}, \Gamma^\theta_{x\theta}$. Compute these Christoffel symbols, and write down the equation satisfied by a geodesic $\gamma(t) = (x(t), \theta(t))$ explicitly.

Part II

Do **three** of the following five problems.

(1) Compute the fundamental group of the orientable surface of genus 2. Recall that this surface can be thought of as taking two tori, removing a small disk from each, and gluing them along the boundary of the disk.

(2) (a) Give an example of a covering map $f : Y \to S^1 \vee S^1$ with deck group S_3 (the symmetric group on 3 letters) and Y path connected.
(b) Define what it means for a covering to be normal (equivalently, Galois), and determine whether the example you wrote down in the previous part is normal.

(3) (a) For a CW complex X, define the cellular (CW) homology of X with integer coefficients.
(b) If X has finitely many cells, then its Euler characteristic can be alternately defined as either

$$\chi(X) = \Sigma(-1)^n c_n(X)$$

where $c_n(X)$ is the number of cells in dimension n, or as

$$\chi'(X) = \Sigma(-1)^n b_n(X)$$

where $b_n = \text{rank}(H_n(X;\mathbb{Z}))$ are the Betti numbers of X. Show that $\chi(X) = \chi'(X)$.

(4) (a) Give an example of a connected topological space X with $\pi_1(X) \cong \mathbb{Z}/6\mathbb{Z}$ and $H_2(X;\mathbb{Z}) = 0$.
(b) For this space X, compute $H_n(X \times X; \mathbb{Z}/2\mathbb{Z})$ for $n = 1$ and $n = 2$.

(5) Let $\mathbb{RP}^n$ be real projective space of dimension n. For this problem, you may only use knowledge of $H_*(\mathbb{RP}^n)$ if you compute it first.
(a) Exhibit a continuous map $S^n \to \mathbb{RP}^n$ which is not homotopic to a constant map.
(b) Compute $H_i(\mathbb{RP}^5; \mathbb{Z}/3\mathbb{Z})$ for $i \geq 0$.

Part III

Do **three** of the following five problems.

(1) Give a complete proof of the homeomorphisms

$$\mathrm{SO}(3) \cong \mathbb{RP}^3 \text{ , and}$$
$$\mathrm{SO}(4) \cong S^3 \times \mathbb{RP}^3 \text{ .}$$

You may cite any *standard* results from topology or linear algebra, so long as those results are clearly stated. (E.g., don't write "from linear algebra, $\mathrm{SO}(3) \cong \mathbb{RP}^3$").

(2) Consider the 2-torus with its standard orientation. The Heisenberg manifold M is the total space of an orientable circle bundle over T^2 whose Euler class is the positive generator of $H^2(T^2;\mathbb{Z})$. Compute the cohomology ring $H^*(M;\mathbb{R})$.

(3) Let $G_2(\mathbb{R}^4)$ be the Grassmannian of 2-planes in $\mathbb{R}^4$. Compute the cohomology ring $H^*(G_2(\mathbb{R}^4);\mathbb{R})$.

(4) Consider the quaternionic Stiefel manifold $V_3(\mathbb{H}^n)$ of 3-frames in $\mathbb{H}^n$. Compute the cohomology ring $H^*(V_3(\mathbb{H}^n);\mathbb{R})$.

(5) Let M be an oriented n-manifold with orientation $\mu \in H^n(M;\mathbb{R})$. Recall that a map $f\colon M \to M$ is *orientation-reversing* if $f^*\mu = -\mu$. Prove there is an orientation-reversing homeomorphism

$$f\colon \mathbb{CP}^n \to \mathbb{CP}^n$$

if and only if n is odd.