GEOMETRY/TOPOLOGY PRELIMINARY EXAMINATION, SEPTEMBER 2025

INSTRUCTIONS:

- There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with the highest scores will count.
- In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I

Do **three** of the following five problems.

Problem 1. State the definition of a submersion $F : M \to N$ between smooth manifolds, and show that any submersion is an open mapping: for any open set $U \subset M$, $F(U)$ is open in N.

Problem 2. Let $\mathcal{L}_X$ be the Lie derivative operator on $\Omega^\bullet(M)$, and i_X the interior product operator. Show that for any differential form ω we have

$$(\mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X)\omega = i_{[X,Y]}\omega.$$

Problem 3. Let M be an $n = p + q$ dimensional compact manifold with boundary, and $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$ differential forms on M with α closed, β exact and with

$$\int_M \alpha \wedge \beta \neq 0.$$

Show that there can exist no $(q-1)$-form λ on M such that $\beta = d\lambda$ and $\lambda\big|_{\partial M} = 0$.

Problem 4. Consider the vector fields

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad , \quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Find local coordinates (u, v) in a neighborhood of $(x, y) = (1, 0)$ such that in these coordinates we have

$$X = \frac{\partial}{\partial u} \quad , \quad Y = \frac{\partial}{\partial v}.$$

Is it possible to do the same for the vector fields

$$X' = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad , \quad Y' = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \quad ?$$

(Justify the answer with a construction/proof.)

Problem 5. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇, and let X, Y, Z be any three vector fields on M. Show that

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y).$$

Part II

Do **three** of the following five problems.

Problem 1. A square $ABCD$ is folded up into a torus by identifying opposite edges, so that each edge of the square corresponds to a circle S^1. The space X is formed by attaching two Möbius bands to the torus, one for each of the edges AB and BC, by identifying the boundary of each Möbius band with the circle corresponding to that edge. Compute $\pi_1(X)$.

Problem 2. Give an explicit example of a 4-sheeted connected covering space $f : X \to S^1 \vee S^1$.

Problem 3. Consider the inclusions $\mathbb{RP}^3 \subset \mathbb{RP}^5$ of real projective spaces.

(1) Compute the map induced on $H_*(-;\mathbb{Z})$ by the inclusion.
(2) Let X be the topological space obtained by gluing two copies of $\mathbb{RP}^5$ along $\mathbb{RP}^3$. Compute $H_*(X;\mathbb{Z})$ for $* \geq 0$.

Problem 4.

(1) Give an explicit counterexample to the following statement: for any two finite CW complexes X and Y, we have isomorphisms

$$H_n(X \times Y;\mathbb{Z}) \cong \bigoplus_{i+j=n} H_i(X;\mathbb{Z}) \otimes H_j(Y;\mathbb{Z})$$

for all n.
(2) Show, however, that the above statement does hold under the additional assumption that X only has cells in even dimensions.

Problem 5. Show that there is no continuous map $S^2 \to \mathbb{RP}^2$ inducing an isomorphism on $H_2(-;\mathbb{Z}/2)$.

Part III

Do **three** of the following five problems.

Problem 1. Let M and N be connected closed n-manifolds which are oriented, with fundamental classes $[M]$ and $[N]$. Recall that the *degree* of a map $f : M \to N$ is the integer $\deg(f)$ for which there is an the equality $f_*[M] = \deg(f)[N]$.

(1) Let $g : M \to N$ be a k-sheeted covering. Prove $\deg(g) = \pm k$.
(2) If f has degree ± 1, prove that the map $\pi_1(f) : \pi_1 M \to \pi_1 N$ is surjective.

Problem 2. Prove that an even-dimensional sphere S^{2n} cannot be a topological group for $n > 0$. Make clear in your argument where it fails for an odd-dimensional sphere, such as S^1 or S^3. (Hint: assume that there exists such a map $m : S^{2n} \times S^{2n} \to S^{2n}$, and analyze the ring homomorphism m^*.)

Problem 3. Embed the oriented surface Σ_g of genus g in $\mathbb{R}^3$ in a standard way, and let M_g be the compact 3-manifold bounded by Σ_g. Denote the double of M_g,

$$D := M_g \underset{\Sigma_g}{\cup} M_g \ .$$

Compute the cohomology ring $\mathrm{H}^*(D, \mathbb{Z})$.

Problem 4. Recall that a *good cover* of a manifold $\mathcal{U} = \{U \subset M\}$ is a cover in which every finite intersection of opens $U \in \mathcal{U}$ is either empty or contractible.

(1) Construct a good cover of $\mathbb{RP}^n$.
(2) Prove that any good cover of $\mathbb{RP}^n$ has at least $n + 1$ elements.

Problem 5. Let Σ_g be an orientable surface of genus g, and let $N_h = \mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2$ be a connect sum of h copies of $\mathbb{RP}^2$. Compute the cohomology ring $\mathrm{H}^*(\Sigma_g \# N_h, \mathbb{Z})$.