Algebraic Topology Preliminary Exam
June 2009

Do all of the following questions.

1. Let X be a path-connected space with abelian fundamental group A and contractible universal cover. Suppose $f : Y \to X$ is a continuous map so that

$$0 = f^* : H^1(X, A) \longrightarrow H^1(Y, A).$$

Prove f is null-homotopic.

2.i.) Let X be a CW complex and suppose that for all non-negative integers n, X has only finitely many n-cells. Prove that if $\tilde{H}_*(X, \mathbb{F}_p) = 0$ for all primes p, then $\tilde{H}_*(X, \mathbb{Q}) = 0$.

ii.) Show by example that the hypothesis on the number of cells in X is necessary.

3.i.) Let M be a compact, connected manifold without boundary of even dimension $2n$. Show that there is a unique class $v \in H^n(M, \mathbb{F}_2)$ so that

$$v \smile u = u^2$$

for all $u \in H^n(M, \mathbb{F}_2)$.

ii.) Find v if $M = \mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$.

4. One form of the Poincaré conjecture states that if a smooth 3-manifold M is homotopy equivalent to S^3, then M is diffeomorphic to S^3.

i.) Assuming this version of the Poincaré conjecture, show that the universal cover of any connected 3-manifold M is either S^3 or contractible.

ii.) Now assume M is orientable. Show that M is parallelizable; that is, there exists a trivialization of the tangent bundle, $TM \cong M \times \mathbb{R}^3$.

Over

5. Let X and Y be two spaces and define

$$X * Y = X \times Y \times [-1,1]/\sim$$

where $\sim$ is the smallest equivalence relation so that

$$(x,y,-1) \sim (x',y,-1) \qquad \text{and} \qquad (x,y,1) \sim (x,y',1).$$

Show that there is a short exact sequence

$$0 \to \tilde{H}_{n+1}(X * Y) \longrightarrow \tilde{H}_n(X \times Y) \xrightarrow{p} \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \to 0$$

with $p(a) = (p^1_*(a), p^2_*(a))$. Here p^i are the two projections from the product.

6. Let $X = S_1 \cup S_2 \subseteq \mathbb{R}^3$ be the union of two spheres of radius 2, one about $(1,0,0)$ and the other about $(-1,0,0)$. Thus

$$S_1 = \{(x,y,z) \mid (x-1)^2 + y^2 + z^2 = 4\}$$

and

$$S_2 = \{(x,y,z) \mid (x+1)^2 + y^2 + z^2 = 4\}.$$

i.) Give a description of X as a CW complex.

ii.) Write out the cellular chain complex of X.

iii.) Calculate $H_*(X,\mathbb{Z})$.

7. Consider the tautological $\mathbb{H}$-line bundle on quaternionic projective space, $E \to \mathbb{H}P^n$. For $n=1$, E thus defines a 4-dimensional real vector bundle over $\mathbb{H}P^1 \cong S^4$. Compute the value of the pairing $\langle p_1(E), [S^4]\rangle$. Show whether or not E is equivalent to the tangent bundle of S^4.