Geometry and Topology Preliminary Examination
Northwestern University
Fall 2014

Solve two problems from each group. If you do more than this, your best two scores of each group will count.

All manifolds are assumed to be connected.

Group I

1) Write down an explicit embedding of the fundamental group of the sphere with three handles to the fundamental group of the sphere with two handles as a normal subgroup with quotient $\mathbb{Z}/2\mathbb{Z}$.

2) Let X be the topological space obtained from $\mathbb{R}^3$ by deleting a circle together with a chord. Prove that X is homotopy equivalent to $S^2 \vee S^1 \vee S^1$.

3) Construct a covering space $Y \to S^1 \vee S^1 \vee S^1 \vee S^1$ with the group of deck transformations equal to $(\mathbb{Z}/2\mathbb{Z})^2$. Compute $\pi_1(Y, y_0)$ where y_0 is a pre-image of the base point of $S^1 \vee S^1 \vee S^1 \vee S^1$.

Group II

1) Let $\mathcal{D}$ be the distribution on $\mathbb{R}^3$ spanned by the vector fields

$$X = ye^x \frac{\partial}{\partial y} - \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial x}.$$

(a) Write down an integral submanifold for $\mathcal{D}$ passing through $(1, 0, 0)$

(b) Is $\mathcal{D}$ integrable?

(c) Can you write down an integral submanifold for $\mathcal{D}$ passing through $(1, 1, 1)$?

2) Let M be a connected smooth manifold. Show that the action of the diffeomorphism group of M on M is transitive, i.e. show that given any two points $p, q \in M$ there is a diffeomorphism $F : M \to M$ with $F(p) = q$.

3) Let S^n be the n-dimensional sphere.

(a) Show that there is no nonvanishing smooth vector field on S^n for $n \geqslant 2$ even.

(b) Show that there is a nonvanishing smooth vector field on S^n for $n \geqslant 1$ odd.

(there are more problems overleaf)

Group III

1) Let $A \subset X$ be a subspace of a topological space. Show that $H^1(X, A; \mathbb{Z})$ is torsion-free.

2) Let M be a compact orientable manifold of dimension $2k$, $k \in \mathbb{N}$.

(a) Show that the parity of the Euler characteristic $\chi(M)$ is equal to the parity of $\dim_{\mathbb{R}} H^k(M; \mathbb{R})$.

(b) Assume now that k is odd, and show that $\dim_{\mathbb{R}} H^k(M; \mathbb{R})$ (and hence $\chi(M)$) is even.

(c) Find a compact orientable 4-manifold M such that $\chi(M)$ is odd.

3) Let M_1, M_2 be two compact n-manifolds and for each $j = 1, 2$ let $B_j \subset M_j$ be open subsets with $\psi_j : B_j \to B_1(0) \subset \mathbb{R}^n$ diffeomorphisms. Let $B'_j = \psi_j^{-1}(\overline{B_{1/2}(0)})$ and $M'_j = M_j \backslash B'_j$. Let the *connected sum of* M_1 *and* M_2 be

$$M_1 \sharp M_2 = (M'_1 \sqcup M'_2)/\sim,$$

where the equivalence relation $\sim$ identifies $x_1 \in B_1 \backslash B'_1$ with $x_2 \in B_2 \backslash B'_2$ iff $\psi_1(x_1) = \psi_2(x_2)$. We have that $M_1 \sharp M_2$ is also a compact n-manifold. Show that the Euler characteristic of $M_1 \sharp M_2$ is given by

$$\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n).$$