Geometry Prelim Exam
June 2009

Choose any **three** of the following five questions.

1. Define what it means for a connection on a Riemannian manifold to be symmetric and compatible with the metric. Show that a Riemannian manifold has a unique connection with these properties.

2. (a) Suppose that $\gamma(t)$ is a geodesic in a complete Riemannian manifold. Let $\alpha(s,t)$ be a variation of γ through geodesics, i.e α is a smooth function such that $\alpha(0,\cdot) = \gamma$ and $\alpha(s,\cdot)$ is a geodesic for each fixed s. Show that the vector field $J(t) = \partial\alpha/\partial s(0,t)$ satisfies the Jacobi equation

$$J''(t) + R(J'(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0,$$

where $\dot{\gamma}(t)$ denotes the tangent vector field to the geodesic γ and $'$ denotes covariant differentiation along γ.

   (b) Show that if a Riemannian manifold has negative sectional curvatures, then $\|J(t)\|^2$ is a (not necessarily strictly) convex function of t for any Jacobi field J.

3. Consider the upper half plane $\{(x,y) \in \mathbf{R}^2 : y > 0\}$ with the Riemannian metric such that the length of a vector based at the point (x,y) is $1/y$ times the usual Euclidean length of the vector.

   (a) Show that vertical lines (i.e. those along which the x-coordinate is constant) are geodesics.

   (b) Show that the curvature of the metric is -1 everywhere.

4. (a) Define what it means for a smooth map $f : M \to N$ between two smooth manifolds to be (i) an immersion, (ii) a submersion, and (iii) an embedding.

   (b) Show that if M is compact and f is an injective immersion, then f is an embedding.

Choose any **two** of the following four questions.

5. (a) Define principal G-bundle.

   (b) If M is simply connected and G is a finite group, what are the principal G-bundles over M.

   (c) Classify principal $U(1)$ bundles over S^2.

   (d) Classify principal $U(1)$ bundles over the torus $\mathbb{T}$.

6. Suppose $m < n$ are positive integers. Prove that there is no holomorphic submersion $\pi : \mathbb{CP}^n \to \mathbb{CP}^m$.

7. Suppose Σ is a two dimensional oriented real manifold with boundary S^1. Let $\tilde{\Sigma} = \Sigma \sqcup_{S^1} D$ be the closed surface obtained by gluing in a disc D along S^1.

   (a) Show that $T\Sigma$ is the trivial bundle.

   (b) Suppose $s \in \Omega^0(T\Sigma)$ is a non-vanishing section. We have the canonical trivialization $TD = D \times \mathbb{C}$ which restricts to the glued boundary S^1. Let $f = s|_{\partial\Sigma} : S^1 \to \mathbb{C}$ be the restriction of s to the boundary relative this trivialization. Compute

$$\frac{1}{2\pi i} \int_{S^1} \frac{df}{f}$$

8. (a) State the Maurer-Cartan formula.

   (b) Prove the intuitive fact that if a connection is preserved by the flow of a vector field X, then so is the curvature.