Math 441/2 Preliminary Exam **September 13, 2000**

Do all six problems.

1. Prove any continuous map $f : RP^2 \to S^1 \times S^3$ is homotopic to a constant map.

2. a) Let M be a connected, compact, n-manifold without boundary, $n \geq 2$. Suppose that M cannot be oriented. Show that $H_{n-1}(X, \mathbb{Z}/2\mathbb{Z}) \neq 0$.

b) Show any simply-connected compact n-manifold without boundary, $n \geq 2$, is orientable.

3. Let T_n be an n-holed torus with a chosen orientation. Are the following statements true or false? (If true, supply an example, if false, give an argument.)

a) There exists a degree 1 map $T_1 \to T_2$.

b) There exists a degree 1 map $T_3 \to T_2$.

Recall that a "degree 1 map" takes the orientation class of the source to the orientation class in the target.

4. Let $\Sigma \subseteq \mathbb{R}^2$ be a subspace homeomorphic to S^1. Then, by the Jordan Curve Theorem, $\mathbb{R}^2 - \Sigma$ is the disjoint union of subspaces U and B with U unbounded and B bounded. Furthermore $B \cup \Sigma$ is homeomorphic to the disk D^2.
Let $x \in \mathbb{R}^2 - \Sigma$ and

$$i_* : H_1(\Sigma) \to H_1(\mathbb{R}^2 - \{x\})$$

be the homomorphism induced by inclusion. Prove:

a) If $x \in U$, then $i_* = 0$.

b) If $x \in B$, then i_* is an isomorphism.

5. Let N be a compact manifold without boundary of dimension $n \geq 1$. Let $x_0 \in N$ be a fixed element. Show that the two maps $i_1, i_2 : N \to N \times N$ given by

$$i_1(y) = (x_0, y) \qquad \text{and} \qquad i_2(y) = (y, x_0)$$

are not homotopic.

6. Let $N = \mathbb{C}P^2 - D$ where D is an open disk with the property that the boundary of N is diffeomorphic to S^3. Define a new manifold $M = N \cup_{S^3} N$ where we have identified the two boundary S^3s via an *orientation-reversing* diffeomorphism.

a) What is the integral cohomology ring $H^*(M, \mathbb{Z})$?

b) Would your answer be different if we used an orientation preserving diffeomorphism?