Algebraic Topology Preliminary Exam
September 2009

Do all of the following questions. Homology and cohomology have integer coefficients unless otherwise specified; $\mathbb{C}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^n$ are complex and real projective space, respectively.

1. Let X be the space obtained from a torus $T = S^1 \times S^1$ be attaching a Möbius band M by a homeomorphism from the boundary circle of M to $S^1 \times \{x_0\} \subseteq T$. Compute $\pi_1(X, x_0)$.

2. Let $p : \tilde{X} \to X$ be the universal cover of a path-connected and locally-path connected space X and let $A \subseteq X$ be a path-connected and locally path-connected subspace. Let $\tilde{A}$ be a path component of $p^{-1}(A)$. Show that $\tilde{A} \to A$ is a covering space and that the image of

$$\pi_1(\tilde{A}, b) \to \pi_1(A, a)$$

is the kernel of $i_* : \pi_1(A, a) \to \pi_1(X, a)$. Here $b \in \tilde{A}$ is any basepoint and $a = p(b)$.

3. Let M be a compact and orientable 3-dimensional manifold without boundary. Suppose $H_1 M$ is a torsion group. Show that $H_2 M = 0$.

4.a) Show that $S^3 \times \mathbb{C}\mathrm{P}^\infty$ and $(S^1 \times \mathbb{C}\mathrm{P}^\infty)/(S^1 \times \{x_0\})$ have isomorphic cohomology rings.

b) Show that these two spaces are not homotopy equivalent.

5. Let X be a topological space which can be written as

$$X = U_1 \cup U_2 \cup U_3$$

with each U_i open in X. Suppose U_i and $U_i \cap U_j$ are contractible for $1 \leq i, j \leq 3$. Show

$$\tilde{H}_n X \cong \tilde{H}_{n-2}(U_1 \cap U_2 \cap U_3).$$

Over

6. A topological group is a topological space G with a distinguished point $e \in G$ (the identity) equipped with continuous maps

$$\mu : G \times G \to G$$

and $(-)^{-1} : G \to G$ so that, with these maps, G is a group. For example, the spaces $\mathbb{R}\mathrm{P}^1 = S^1$ and $\mathbb{R}\mathrm{P}^3 = SO(3)$ are topological groups.

Show that if $\mathbb{R}\mathrm{P}^n$ can be given the structure of a topological group, then either $n = 2^t - 1$ for some integer t or $n = \infty$.

7. Let X be the topological space obtained as the quotient of the sphere S^2 under the equivalence relation $x \sim -x$ for x in the equatorial circle.

a) Describe a CW complex whose underlying space is X.

b) Write down the CW chain complex of X.