Topology Prelininary Exam September 8, 1995

Part A: Do TWO of the following 3 problems. (15 points each.)

1. Let X be a compact embedded C^∞ submanifold of $\mathbf{R}^n$ with codimension at least one. Let NX be the normal bundle of X consisting of all pairs $(x, v) \in X \times \mathbf{R}^n$ such that v is perpendicular to all tangent vectors to X at x. Define $f : NX \to \mathbf{R}^n$ by $f(x, v) = x + v$. Show that the derivative of f at $(x, 0)$ is a linear isomorphism for each $x \in X$. Then show that there is an $\varepsilon > 0$ such that the restriction of f to $\{(x, v) \in NX : \|v\| < \varepsilon\}$ is a diffeomorphism onto its image. (You may use the fact that NX is a C^∞ manifold of dimension n.)

2. Suppose that X and Y are C^∞ manifolds, X is compact, and $f : X \to Y$ is a C^∞ map that is transversal to a compact C^∞ submanifold Z of Y. Show that if $F : X \times [0, 1] \to Y$ is a C^∞ map with $F(x, 0) = f(x)$ for all $x \in X$, then the map f_s defined by $f_s(x) = F(x, s)$ is transversal to Z for all small enough $s > 0$.

3. Suppose X is a compact C^∞ manifold, Y is a connected C^∞ manifold with the same dimension as X, and $f : X \to Y$ is a C^∞ map whose mod 2 degree is nonzero. Prove that f is onto.

Part B: Do TWO of the following 3 problems. (15 points each.)

4. Define the cellular chain complex of a CW complex (in particular, define the boundary map of this complex). Provide the 2-torus $S^1 \times S^1$ with a CW structure and exhibit the associated cellular chain complex. Compute the homology and cohomology of this chain complex (and thus of $S^1 \times S^1$).

5. Let X be a finite simplicial complex. Give several different (but equivalent) definitions of the Euler characteristic and verify that each gives the same number for X.

6. Give the general construction of the covering space of a connected, pointed, locally path connected space X, x associated to a subgroup of $\pi_1(X, x)$. Make this more specific by repeating this construction for the explicit example of $X = S^1 \vee S^1$ and the subgroup $\mathbf{Z} \cdot s \subset \mathbf{F}(s, t) = \pi_1(S^1 \vee S^1)$, where $\mathbf{F}(s, t)$ denotes the free group on generators s, t. Reinterpret "geometrically" (i.e., in terms of covering spaces) the fact that this subgroup is not normal.

Part C: Do EACH of the following 3 problems. (10 points each.)

7. Give examples of each of the following and *briefly* justify each example:

(a) Two spaces which are homotopy equivalent but not homeomorphic.
(b) Two connected pointed spaces X, Y with $H_1(X, \mathbf{Z}) \cong H_1(Y, \mathbf{Z})$, $\pi_1(X, x) \not\cong \pi_1(Y, y)$.
(c) A CW complex X such that $H_1(X, \mathbf{Z})$ is an uncountable free abelian group.
(d) A topological space which is not a CW complex.

8. Use Mayer-Vietoris to compute the integral homology of the following spaces:

(a) The subspace of $\mathbf{R}^3$ given as the union of the unit sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and the unit 2-disk $D^2 = \{(x, y, 0) : x^2 + y^2 \leq 1\}$.
(b) The connected sum of a torus and a Klein bottle.

9. Prove that the *singular* homology of a 2-dimensional simplicial complex X vanishes in dimensions greater than 2 (i.e., $H_i(X, \mathbf{Z}) = 0,\ i > 2$).