Topology Preliminary Exam, September 1996

Do 2 of 3 problems from Part A and 4 of 6 problems from Part B.

Be sure to indicate which problems you are submitting.

Part A

In **Part A**, X will be the subset of $\mathbf{R}^3$ defined by $x^2 + y^2 + z^2 = 1$.

(A1.) (a) Show that X is a smooth manifold.

(b) Describe the normal space to X (as a subset of $\mathbf{R}^6$).

(A2.) (a) Describe the tangent space to X (as a subset of $\mathbf{R}^6$).

(b) Calculate the self intersection number of X.

(A3.) Consider the function $f : X \to \mathbf{R}^4$ given by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Let Y be the image set $f(X)$.

(a) Show that Y is a smooth manifold and that the map f is smooth.
(b) Describe the tangent space to Y.
(c) Calculate the mod 2 self intersection number of Y.

Part B

(B1.) Let $p : \tilde{X} \to X$ be a covering map. Assume $\tilde{X}$ and X are path connected and that $\pi_1(X, x_0)$ is finite. Let $\tilde{x}_0 \in \tilde{X}$ so that $p(\tilde{x}_0) = x_0$. Show that the number of points in $p^{-1}(x_0)$ is equal to $[\pi_1(X, x_0) : p_*\pi_1(\tilde{X}, \tilde{x}_0)]$, the index of the image of p_* in the fundamental group of X.

(B2.) Let $f : S^3 \to S^3$ be a map such that $f(-x) = -f(x)$ for all $x \in S^3$. Then f induces a map $g : \mathbf{R}P^3 \to \mathbf{R}P^3$.

(a) Show that $g_* : \pi_1(\mathbf{R}P^3, *) \to \pi_1(\mathbf{R}P^3, *)$ is an isomorphism.

(b) Show that $f_* : H_3(S^3; \mathbf{Z}) \to H_3(S^3; \mathbf{Z})$ is multiplication by an odd integer.

(B3.) (a) Describe a map $f : S^2 \times S^2 \to S^4$ such that $f_* : H_4(S^2 \times S^2; \mathbf{Z}) \to H_4(S^2; \mathbf{Z})$ is an isomorphism.

(b) Does there exist a map $f : S^4 \to S^2 \times S^2$ such that $f_* : H_4(S^2; \mathbf{Z}) \to H_4(S^2 \times S^2; \mathbf{Z})$ is an isomorphism? Explain your answer.

(B4.) Give an example of a map $f : S^p \to S^q$, for some p and q with $p > q$, such that f is not homotopic to a constant map. Include a proof that your map f is, in fact, not homotopic to a constant map.

(B5.) Let P be the projective plane and let K be the Klein bottle.

(a) What are $H_*(K; \mathbf{Z})$ and $H^*(K; \mathbf{Z})$?

(b) What is $H_*(K \times P; \mathbf{Z})$?

(B6.) Let M be a compact connected manifold of dimension m and let N be a compact connected manifold of dimension n, both without boundary. Suppose there is a map $f : M \to N$ which is one-to-one.

(a) Show that $m \leq n$.

(b) Show that if $m = n$, then f is a homeomorphism of M onto N.