Topology Preliminary Exam, September 1997

Do 2 of 3 problems in Part A and 4 of 6 problems in Part B

Be sure to indicate which problems you are submitting.

Part A

A1. Define *chart, atlas* and *manifold.* Define C^∞ *function* from one manifold to another and define *diffeomorphism.*

If $f : M \to N$ is a C^∞ function from the manifold M to the manifold N prove that its graph $G = \{(x, f(x)) \in M \times N |\ x \in M\}$ has the structure of a manifold and that it is diffeomorphic to M.

A2. Define *critical point* and *critical value* of a smooth function. State Sard's Theorem. Consider two C^∞ functions $f : \mathbf{R} \to \mathbf{R}$ and $g : S^1 \to S^1$. Is it possible that the set of critical values of f are dense in $\mathbf{R}$ or that the set of critical values of g are dense in S^1? In each case prove your answer; i.e. either prove it is not possible or construct an example.

A3. Define *transversal intersection* of two submanifolds.

Suppose M^m and N^n are compact manifolds of dimension m and n respectively. Let $f : M \to N$ be a C^∞ function and suppose $y \in N$ is a regular value of f. Let $W = f^{-1}(y)$ so that W is a submanifold of M. Assume W is non-empty. If V is a submanifold of M which has a non-empty transverse intersection with W then give upper and lower bounds (in terms of m and n) on the possible dimension of V. Justify your answer. Give an example to show that at least one of these bounds is not valid if $W = f^{-1}(y)$ is a submanifold of M but $y \in N$ is not a regular value of f.

Part B

B1. A space Y is said to *dominate* a space X if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map of X. Suppose that X and Y are *finite* CW-complexes such that both X dominates Y and Y dominates X. Prove that $H_p(X) \approx H_p(Y)$ for all $p \geq 0$.

B2. Describe a CW-complex structure for the complex projective plane CP^2 *including a description of the attatching maps of the cells*. In addition, prove that CP^2 is simply-connected.

B3. Suppose that $f : S^4 \to S^4$ is a map such that $f(-x) = -f(x)$. Then f induces a map $g : RP^4 \to RP^4$ by $g(\{x, -x\}) = \{f(x), -f(x)\}$. Prove that $g^* : H^1(RP^4; \mathbf{Z}/2\mathbf{Z}) \to H^1(RP^4; \mathbf{Z}/2\mathbf{Z})$ is an isomorphism.

Hint: First prove that g induces an isomorphism of the fundamental group.

B4. What are the following homology and cohomology groups (for all values of p)?

$$H_p(RP^3; \mathbf{Z}),\ H_p(RP^3; \mathbf{Z}/2\mathbf{Z}),\ H^p(RP^3; \mathbf{Z}/5Z),\ H_p(RP^3 \times RP^2; \mathbf{Z})$$

B5. The diagonal $D = \{(x, x) \in S^1 \times S^1 \mid x \in S^1\}$ of $S^1 \times S^1$ is homeomorphic to S^1. Suppose that two copies of $S^1 \times S^1$ are glued together along the diagonal D to obtain the space Y. Calculate the homology groups of Y.

Remark: D has a neighborhood in $S^1 \times S^1$ homeomorphic to an open annulus $S^1 \times (-1, 1)$.

B6. Calculate the cohomology *ring* $H^*(K; \mathbf{Z}/2\mathbf{Z})$. of the Klein bottle K.

Hint: K is the connected sum of two copies of RP^2 and has a useful map onto $RP^2 \vee RP^2$.