Preliminary Examination **Algebraic Topology** **February 1999**

Part A: Do TWO of the following 3 problems.

A1. (a) Define *chart, atlas* and *manifold.*

(b) Let X denote the one point compactification of the complex numbers C. So $X = C \cup \{\infty\}$ and the sets $\{z : |z| > r, r > 0\} \cup \{\infty\}$ form a neighborhood basis of $\{\infty\}$. Prove that X is a manifold and the function $f : X \to X$ is smooth if f is defined by $f(z) = z^2$ if $z \in C$ and $f(\infty) = \infty$. In particular prove f is smooth at the point ∞.

A2. (a) Define *transversal intersection* of two submanifolds.

(b) Suppose $f : R^n \to R$ and $g : R^n \to R$ are C^∞ functions and that the derivatives $df_x \neq dg_x$ whenever $f(x) = g(x)$. Prove that the graphs of f and g are submanifolds of R^{n+1} and that they intersect transversally. Note: the graph of f is $\{(x, f(x)) \in R^{n+1} : x \in R^n\}$.

A3. (a) Define *critical point* and *critical value* of a smooth function. State Sard's Theorem.

(b) Suppose $f : S^1 \to R^4$ is a smooth embedding. Prove that there is a three dimensional subspace V of R^4 such that $P \circ f : S^1 \to V$ is one-to-one, where P is orthogonal projection of R^4 onto V.

Part B: Do EACH of the following 3 problems.

B1. Calculate $H_p(RP^3 \times RP^3; \mathbf{R})$ for all p and for $\mathbf{R} = \mathbf{Z}/2\mathbf{Z}$ and $\mathbf{R} = \mathbf{Z}/3\mathbf{Z}$.

B3. A homology class $\alpha \in H_n(X; \mathbf{Z})$ is called *spherical* if there is a map $f : S^n \to X$ such that $f_*(\mu) = \alpha$, where μ generates $H_n(X; \mathbf{Z})$.

Which classes $\alpha \in H_{p+q}(S^p \times S^q; \mathbf{Z})$ are spherical $(p \geq 1, q \geq 1)$?

B3. Calculate $H_*(X; \mathbf{Z})$ where $X = S^2 \cup \{(0, 0, t) \in \mathbf{R}^3 \mid -1 \leq t \leq 1\} \cup (D^2 \times \{0\})$.

In words: X is the union of a 2-sphere with an equatorial disk and with a line segment joining the North and South poles.

Part C: Do TWO of the following 4 problems.

C1. Can $\mathbf{C}P^2$ be homeomorphic to a proper subspace of itself? Explain your answer.

C2. Let $p : X \to Y$ be a covering map with X (and hence Y) path-connected and locally path-connected. Suppose there is a map $f : X \to X$ such that $p \circ f = p$. Show that f is a homeomorphism.

C3. Suppose $A \subset X$ where X is contractible. Suppose that $\alpha \in H^p(X, A)$ and $\beta \in H^q(X, A)$ where $p > 0$ and $q > 0$. Show that $0 = \alpha \cup \beta \in H^{p+q}(X, A)$.

C4. Prove that S^{2n} cannot be a covering space of $\mathbf{C}P^n$ if $n \geq 2$.