**Math 441/2 Preliminary Exam**

September 2002

Do as many of the following as possible. As a matter of strategy it is better to do one question completely correct rather than two partly correct, but it is better to do something rather than nothing on any question. Four correct answers and two partially correct answers will be a clear pass.

1. Let $f : M \hookrightarrow N$ be a $C^\infty$ embedding of differentiable manifolds of dimension $m$ and $n$ respectively. Suppose $m < n$. Let $x \in M$. Show that there exists a neighbourhood $U \subset N$ of $f(x)$ and functions $g_i \in C^\infty(U)$, $1 \leq i \leq n - m$, such that

$$f[M] \cap U = \{x \in U \mid g_i(x) = 0,\ 1 \leq i \leq n - m\}.$$

2. Let $\mathbb{R}\mathrm{P}^k$ be the real projective space of dimension $k$. Then $\mathbb{R}\mathrm{P}^1 = S^1$ and

$$\mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1 \cong S^2.$$

i.) Show that $H_2(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \to H_2(S^2, \mathbb{Z})$ is zero, but that $H_2(\mathbb{R}\mathrm{P}^2, \mathbb{Z}/2) \to H_2(S^2, \mathbb{Z}/2)$ is non-zero.

ii.) Prove or give a counterexample to the following statement: the isomorphism

$$H_n(X, A) \cong H_n(X) \otimes A \oplus \mathrm{Tor}(H_{n-1}(X), A)$$

given by the universal coefficient theorem is natural.

3. Let $\tilde{H}_*(X)$ be the reduced homology (over the integers) of $X$. Give an example of two spaces $X$ and $Y$ so that

$$\tilde{H}_*(X) \neq 0 \neq \tilde{H}_*(Y)$$

and

$$\tilde{H}_*(X \times Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y).$$

4. Let $M$ be a closed, connected, orientable $n$-manifold with $n \geq 1$. Suppose that the product of any two elements of positive degree in $H^*(X, \mathbb{Q})$ is zero. Argue that there is an isomorphism of cohomology rings

$$H^*(M, \mathbb{Q}) \cong H^*(S^n, \mathbb{Q}).$$

**Over**

5. a) Let $M$ be a closed $n$-manifold. Choose a point $x \in M$ and let $D$ be an open ball around $x$ contained in a coordinate chart. Define $M_0 = M - D$. The inclusion of the boundary of $D$ defines a homeomorphism of $S^{n-1}$ to the boundary of $M_0$. Calculate the graded vector space

$$H_*(M_0, \mathbb{Z}/2).$$

b) If $M$ and $N$ are two such $n$-manifolds, define their connected sum by the formula

$$M \sharp N = M_0 \cup_{S^{n-1}} N_0.$$

Calculate $H_*(M \sharp N, \mathbb{Z}/2)$ as a graded vector space. Here $S^{n-1}$ is identified with the boundary of both $M$ and $N$.

6. Let $n$ be a positive *even* integer. Consider the composite

$$S^{2n-1} \xrightarrow{f} S^n \vee S^n \xrightarrow{g} S^n$$

where

a) $g$ is the identity on each of the two factors of the one-point union $S^n \vee S^n$ and

b) $f$ is a map so that $(S^n \vee S^n) \cup_f D^{2n} = S^n \times S^n$.

Calculate the cohomology ring $H^*(S^n \cup_{gf} D^{2n}, \mathbb{Z})$. Then decide whether or not the map $gf$ is null homotopic.