Math 441/2 Preliminary Exam

September 2003

Do at least six of the following, and at least one of the first two.

1. Define what it means for a space M to be a C^∞ manifold. Begin with appropriate notions of *chart* and *atlas*. Define what it means for a function $f : M \to N$ between manifolds to be C^∞. Finally prove that if M and N are C^∞ manifolds, so is $M \times N$ and the projection $M \times N \to N$ is C^∞.

2. State Sard's Theorem. If $f : S^1 \to S^1$ is a C^∞ map, is it possible for the set of critical points of f to be dense in S^1?

3. In this problem (co-)homology has integer coefficients. Let $X = S^2 \times S^2$ and consider the diagonal copy of $S^2 \subseteq X$. Calculate the map $H_2(S^2) \to H_2(X)$ induced by the inclusion. Now let $Y = X \cup_{S^2} X$ be the space obtained by gluing two copies of X along the diagonal S^2s. Calculate the cohomology ring $H^*(Y)$.

4. Show that $\mathbb{R}P^2 \times S^2$ and $\mathbb{R}P^4 \vee S^2$ do not have the same homotopy type.

5. Let C be a chain complex over the rational numbers so that C_n is finite dimensional for each n, $C_n = 0$ for $n < 0$, and there is an integer N so that $C_n = 0$ for $n > N$. Show that $\sum_{n=0}^{\infty} \dim_{\mathbb{Q}} C_n = \sum_{n=0}^{\infty} \dim_{\mathbb{Q}} H_n C$.

6. Let M be a simply-connected, closed, compact manifold. Show that M is orientable. To begin, you might determine $H_1(M, \mathbb{Z}/2\mathbb{Z})$.

7. Let $X \subseteq \mathbb{R}^3$ be the union of S^2 and the line segment $L = \{(0,0,t) \mid -1 \leq t \leq 1\}$. Give a description of X as a CW complex, and describe the resulting *CW* chain complex. Calculate the homology of that chain complex.

8. In this problem homology has integer coefficients. A homology class $x \in H^n(X)$ is called *spherical* if there is a continuous map $f : S^n \to X$ so that x is in the image of the induced map in homology. Identify all the spherical classes of positive degree in $H_*(\mathbb{C}P^\infty)$. Identify all the spherical classes of positive degree in $H_*(S^2 \times S^2)$.