Math 441/2 Preliminary Examination
September 2005

Do all of the following questions. All homology and cohomology is with integer coefficients unless otherwise stated.

1.a.) Let X be a compact submanifold of a $\mathcal{C}^\infty$ manifold Y with $\dim Y = 2n$, $\dim X = n$, and both manifolds oriented. Suppose there exist $f_1, \ldots, f_n \in \mathcal{C}^\infty(Y)$ with df_i linearly independent on X such that $X = \{f_i = 0 \text{ for all } i\}$. Show that the intersection number $I(X, X) = 0$.

b.) Use the first part to show that the diagonal submanifold of $S^2 \times S^2$ cannot be defined by the vanishing of two independent $\mathcal{C}^\infty$ functions.

2.a.) Define what it means for a map $f : X \to Y$ of $\mathcal{C}^\infty$ manifolds to be a submersion.

b.) Let X be compact and $f_t : X \to Y$ a smooth family of maps parametrized by $t \in [0, 1]$; that is, $F(t, x) := f_t(x)$ is a smooth map $[0, 1] \times X \to Y$. Let f_0 be a submersion. Show there exists $\epsilon > 0$ such that f_t is a submersion for all $t < \epsilon$.

3. Let $Y = T \cup S^2$ where T is the torus in $\mathbb{R}^3$ obtained by rotating the circle

$$C = \{ \ (x, y) \mid (x - 2)^2 + y^2 = 1 \ \}$$

about the y-axis. Here S^2 denotes the two sphere of radius one centered at the origin.

a.) Compute the homology groups of Y.

b.) Compute the cohomology ring of Y.

4. Let X be a finite CW complex, let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements and let $\mathbb{Q}$ be the field of rational numbers. Prove or disprove the following statement: there is an integer N so that for all primes $p > N$ and all integers $k \geq 0$,

$$\dim_{\mathbb{F}_p} H_k(X, \mathbb{F}_p) = \dim_{\mathbb{Q}} H_k(X, \mathbb{Q}).$$

Here the symbol $\dim_k$ indicates the dimension of the indicated vector space.

5.a.) Calculate the orientation double cover of $\mathbb{R}\mathrm{P}^{2n+1}$; explain why this calculation demonstrates that $\mathbb{R}\mathrm{P}^{2n+1}$ is orientable.

b.) State the Poincare Duality Theorem for compact manifolds without boundary. To do this you will have to first define the notion of an orientation class.

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6.a.) Let Z be a path-connected space so that $H_1(Z) = 0$. What can you say about $\pi_1(Z)$?
b.) Prove or disprove: There exists a space Z satisfying $H_1(Z) = 0 \neq \pi_1(Z)$.

7. Let ξ be an oriented real n-place bundle over a space X. For this problem you may assume the Thom Isomorphism Theorem.

a.) Define the Euler class $e(\xi)$ of ξ.

b.) Prove the existence of a long exact sequence (the Gysin Sequence)

$$\cdots \to H^{k-n}X \xrightarrow{\smile e(\xi)} H^k X \to H^k E(\xi)_0 \to H^{k-n+1}X \to \cdots$$

Here $E(\xi)_0 \subseteq E(\xi)$ is the total space of ξ minus the zero section.

c.) Let $\xi = \gamma$ be the canonical $\mathbb{C}$-plane bundle over $\mathbb{C}\mathrm{P}^n$. Use the Gysin sequence to give a calculation of the cohomology ring $H^*(\mathbb{C}\mathrm{P}^n)$.