**Algebraic Topology Preliminary Exam**
**September 2007**

Do all of the following questions. Homology and cohomology have integer coefficients unless otherwise specified.

1. Let $X$ be the space obtained from a torus $T = S^1 \times S^1$ be attaching a Möbius band $M$ by a homeomorphism from the boundary circle of $M$ to $S^1 \times \{x_0\} \subseteq T$. Compute $\pi_1(X, x_0)$.

2. Let $p : \tilde{X} \to X$ be the universal cover of a path-connected and locally-path connected space $X$ and let $A \subseteq X$ be a path-connected and locally path-connected subspace. Let $\tilde{A}$ be a path component of $p^{-1}(X)$. Show that $\tilde{A} \to A$ is a covering space and that the image of

$$\pi_1(\tilde{A}, b) \to \pi_1(A, a)$$

is the kernel of $i_* : \pi_1(A, a) \to \pi_1(X, a)$. Here $b \in \tilde{A}$ is any basepoint and $a = p(b)$.

3. Let $M$ be a closed orientable 3-dimensional manifold. Write $H_1 M$ as $F \oplus T$ where $F$ is a free abelian group and $T$ is a torsion abelian group. Prove $H_2 M \cong F$.

4. Show that $S^3 \times \mathbb{C}\mathrm{P}^\infty$ and $(S^1 \times \mathbb{C}\mathrm{P}^\infty)/(S^1 \times \{x_0\})$ have isomorphic cohomology rings.

5. Let $X$ be a topological space which can be written as finite union of open subspaces $U_i \subseteq X$, $1 \leq i \leq n$. Suppose for all pairs $i, j$, $1 \leq i \leq j \leq n$, the intersection

$$U_i \cap U_j$$

has the property that all its path components are contractible. Prove $H_m X = 0$ for $m \geq n$.

6. Let $X$ be the Moore space obtained from $S^n$, $n \geq 1$, by attaching an $(n+1)$-cell by a map of degree $d$. Let $f : X \to S^{n+1}$ be the quotient map obtained by collapsing the $n$-sphere.

a.) Show that $0 = f_* : \tilde{H}_* X \to \tilde{H}_* S^{n+1}$, but that $0 \neq f^* : H^{n+1} S^{n+1} \to H^{n+1} X$.

b.) Deduce that the splitting in the Universal Coefficient Theorem for cohomology cannot be natural.

7. Let $X$ be the topological space obtained as the quotient of the sphere $S^2$ under the equivalence relation $x \sim -x$ for $x$ in the equatorial circle. Describe a CW complex whose underlying space is $X$. Compute the homology $H_*(X)$.