

GEOMETRY/TOPOLOGY PRELIMINARY EXAMINATION,
SEPTEMBER 2026

INSTRUCTIONS:

- There are **three** parts to this exam. Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.
- In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I

Do **three** of the following five problems.

- (1) Let M be a compact smooth manifold, and let $f : M \rightarrow \mathbb{R}$ be smooth.
- (a) Explain why for almost every $c \in \mathbb{R}$ the level set $f^{-1}(c)$ is either empty, or a smooth embedded hypersurface in M .
 - (b) Suppose that $a < b$ and f has no critical points in $f^{-1}([a, b])$. Prove that the level sets $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.

- (2) Let α be a nowhere vanishing 1-form on a smooth manifold M .

- (a) Prove the identity

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]).$$

- (b) Show that the distribution $\mathcal{D} = \ker \alpha \subset TM$ is involutive if and only if $d\alpha(X, Y) = 0$ for all $X, Y \in \ker \alpha$.
- (c) Let $\alpha = dz - x dy$ on $\mathbb{R}^3$. Is there a two-dimensional integral manifold of $\mathcal{D}$ passing through every point?

- (3) Let M be a smooth manifold, and ∇ a connection on TM . For $f \in C^\infty(M)$ define

$$\text{Hess}_\nabla f(X, Y) = X(Yf) - (\nabla_X Y)f.$$

- (a) Prove that $\text{Hess}_\nabla f$ is a $(0, 2)$ -tensor.
- (b) Prove that if ∇ is torsion free, then $\text{Hess}_\nabla f$ is symmetric.

- (4) Let $F : M \rightarrow N$ be a smooth map, and let h be a Riemannian metric on N .

- (a) Prove that the symmetric $(0, 2)$ -tensor F^*h is a Riemannian metric if and only if F is an immersion.
- (b) Assuming that F is an immersion, prove that

$$\ell_{F^*h}(\gamma) = \ell_h(F \circ \gamma),$$

for every smooth curve γ in M , where ℓ_g denotes the length of a curve using the Riemannian metric g .

- (5) Let (M, g) be a Riemannian manifold, let $p \in M$, and suppose that for some $R > 0$ the exponential map $\exp_p$ is a diffeomorphism from the ball $B_R(0) \subset T_p M$ to a neighborhood U of p .

Show that for any $v \in B_R(0)$, the radial geodesic $\gamma(t) = \exp_p(tv)$, for $t \in [0, 1]$, has the shortest length among any smooth curve from p to $\exp_p(v)$ that is contained in U .

Part II

Do **three** of the following five problems.

- (1) Let Σ_g be an orientable surface of genus $g \geq 1$.
 (a) Prove that connected two-sheeted covers of Σ_g correspond to nonzero homomorphisms

$$\pi_1(\Sigma_g) \longrightarrow \mathbb{Z}/2.$$

- (b) Show that there are exactly

$$2^{2g} - 1$$

isomorphism classes of connected two-sheeted covers of Σ_g .

- (2) Let X be a path-connected CW complex, and let its suspension be

$$\Sigma X = X \times [0, 1] / \sim$$

where $\sim$ is the relation collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to (another) point. Let $\tilde{H}_*(X; \mathbb{Z})$ be the reduced homology of X with integer coefficients.

- (a) Prove that ΣX is simply connected.

- (b) Prove that

$$\tilde{H}_k(\Sigma X; \mathbb{Z}) \cong \tilde{H}_{k-1}(X; \mathbb{Z})$$

for every k .

- (c) Compute $\pi_1(\Sigma \mathbb{R}P^2)$ and all the integral homology groups of $\Sigma \mathbb{R}P^2$. Is this space homotopy equivalent to a wedge of spheres?

- (3) Consider the spaces

$$X = S^1 \times S^2, \quad Y = S^1 \vee S^2 \vee S^3.$$

- (a) Compute the fundamental groups of X and Y .
 (b) Compute the integral homology groups of X and Y in each degree.
 (c) Describe the universal covers of X and Y .

- (4) For $n \geq 1$, let $f : S^n \longrightarrow S^n$ be a continuous map such that $f(x) = f(-x)$ for every $x \in S^n$ (here, $-x$ denotes the point on the sphere antipodal to x).

- (a) If n is even, show that f has degree 0.
 (b) For every odd $n \geq 1$, construct a map f which has any prescribed even degree.

- (5) Let X be a CW complex with finitely many cells in each degree. Suppose that for any prime p and any $k \geq 0$, we have

$$H_{2k+1}(X; \mathbb{Z}/p) \cong 0.$$

- (a) If n is odd, show that $H_n(X; \mathbb{Z}) \cong 0$.
 (b) If n is even, show that $H_n(X; \mathbb{Z})$ is a free abelian group.

Part III

Do **three** of the following five problems.

- (1) Define the Hopf map $S^3 \rightarrow S^2$. Prove that the map you have defined is not nullhomotopic.

- (2) Give complete proofs of the homeomorphisms

$$\mathrm{SU}(2) \cong S^3, \text{ and}$$

$$\mathrm{U}(2) \cong S^1 \times S^3.$$

- (3) Let S be an oriented compact surface of genus 2 and $\mu \in H^2(S; \mathbb{Z})$ be an orientation class for S . Let M be an oriented circle bundle over S whose Euler class is

$$e = 3\mu.$$

Compute the cohomology groups $H^*(M; \mathbb{Z})$.

- (4) Consider the complexification map

$$c: \mathbb{RP}^2 \rightarrow \mathbb{CP}^2.$$

Compute the integral homology groups $H_*(C; \mathbb{Z})$ of the mapping cone C of the map c , with proofs.

- (5) Consider the quaternionic flag manifold $F_n(\mathbb{H}^n)$ of complete flags in $\mathbb{H}^n$. Compute the cohomology ring $H^*(F_n(\mathbb{H}^n); \mathbb{R})$.