Preliminary Exam in Analysis
June 2025

Instructions:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let f be a measurable function from X to $[0, \infty]$ with $\int_X f < \infty$. Show that for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > \left(\int_X f\right) - \epsilon$.

(2) Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $E_n \in \mathcal{M}$ with $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$, and $\chi_{E_n} \to f$ in measure, where χ_{E_n} is the characteristic function of E_n. Show that f is a.e. equal to the characteristic function of a measurable set.

(3) Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$. Show that $|\nu|(E) = \sup\left\{\left|\int_E f d\nu\right| : |f| \leq 1\right\}$, where $|\nu|$ is the total variation of ν.

(4) Suppose that μ, ν are σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$, $f = \frac{d\nu}{d\lambda}$. Show that $0 \leq f < 1$ μ-a.e. and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.

(5) Suppose $F : \mathbb{R} \to \mathbb{C}$. Show that the following are equivalent:
   (a) There is a constant M such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$.
   (b) F is absolutely continuous and $|F'| \leq M$ m-a.e., where m is the Lebesgue measure.

Part II. Functional Analysis

Note: You may use any (consistent) normalization that you prefer for Fourier transforms and Fourier series. Do **three** of the following five problems.

(1) Let $(X, \mathscr{F}, \mu)$ be a measure space and $1 \le p, q, r \le \infty$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

Show that for all nonnegative measurable functions f, g, h on X,

$$\|fgh\|_1 \le \|f\|_p \|g\|_q \|h\|_r.$$

(2) Let $1 \le p, q \le \infty$ be a pair of conjugate exponents: $1/p + 1/q = 1$. Suppose that f is a (real valued) measurable function on $[0,1]$ with the following property: there is a constant C such that for every $g \in L^q[0,1]$ we have $fg \in L^1[0,1]$ and $\|fg\|_1 \le C\|g\|_q$. Show that $f \in L^p[0,1]$ and $\|f\|_p \le C$.

(3) Let $H = L^2[0,1]$ be the Hilbert space of square integrable functions on $[0,1]$. Define the integral operator $K : H \to H$ by

$$Kf(x) = \int_0^x f(y)\, dy, \quad f \in H.$$

(a) Show that K is a compact operator.
(b) Find the adjoint operator K^*.
(c) Find the integral kernel of the symmetric operator $L = K^*K$ (the composition of K and K^*). Namely, find the function $L(x, y)$ such that

$$Lf(x) = \int_0^1 L(x, y) f(y)\, dy.$$

(4) Let X be a Banach space and L a closed proper subspace of X (i.e., L is not equal to X). For any element $x \in X$ define

$$d(x, L) \stackrel{\text{def}}{=} \inf_{y \in L} \|x - y\|.$$

(a) Suppose that X is a Hilbert space,. Show that there is an element $x \in X$ such that $d(x, L) = 1$ and $\|x\| = 1$.
(b) In general show that for any $\epsilon > 0$, there is an element $x \in X$ such that $\|x\| = 1$ and $d(x, L) \ge 1 - \epsilon$.
(c) Show that an infinite dimensional Banach space cannot be locally compact.

(5) Define $f(\varphi) = (\Delta\varphi)(0)$ for $\varphi \in \mathscr{S}(\mathbb{R}^n)$, where

$$\Delta\varphi = \sum_{i=1}^n \frac{\partial^2 \varphi}{\partial x_i^2}$$

is the Laplace operator.
(a) Show that f is a tempered distribution and find its Fourier transform $\hat{f}$.
(b) For what values of $s \in \mathbb{R}$ do we have $f \in H^s(\mathbb{R}^n)$, the Sobolev space of index s?

Part III. Complex Analysis

Do **three** of the following five problems.

(1) Let $f : D_2(0) \to \mathbb{C}$ be holomorphic, such that there are infinitely many values z for which $|z| = 1$ and $f(z)$ is real. Here $D_2(0) = \{z : |z| < 2\}$.
  (a) Considering the function $z \mapsto \overline{f(1/\bar{z})}$ as well, show that f is real valued on the circle $|z| = 1$.
  (b) Show that f is constant.

(2) Suppose that $f : \Omega \to \mathbb{H}$ is holomorphic, where $\Omega = \{z : |z| > 1\}$, and $\mathbb{H}$ is the upper half plane. Show that the limit $\lim_{z\to\infty} f(z)$ exists (possibly infinite).

(3) Use contour integration to compute the integral

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx.$$

Hint: you can consider a contour integral of the function $\frac{e^{iz}-1}{z(z^2+1)}$.

(4) Let $\Omega = \{z : |z| \neq 0, \quad \arg z \in (0, \pi/2)\}$, and let I denote the closed line segment from 0 to $1+i$.
  (a) Find a biholomorphism from Ω to the unit disk $\mathbb{D}$.
  (b) Find a biholomorphism from $\Omega \setminus I$ to the unit disk.

(5) Let $0 \in \Omega_1 \subsetneq \Omega_2 \subset \mathbb{C}$ be open, and $f_i : \Omega_i \to \mathbb{D}$ biholomorphisms such that $f_i(0) = 0$ for $i = 1, 2$. Show that $|f_1'(0)| > |f_2'(0)|$.