PRELIMINARY EXAM IN ANALYSIS
SEPTEMBER 2024

INSTRUCTIONS:

(1) This exam has **three** parts: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part. If you attempt more than three problems in one part, then the three problems with highest scores will count.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. However, if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) Fix a measure space $(X, \mathcal{M}, \mu)$. Suppose $f_n \geq 0$ and $f_n \to f$ *in measure*. Show that

$$\int_X f \, d\mu \leq \liminf \int_X f_n \, d\mu.$$

(2) Let f be Lebesgue integrable on the interval $(0, a)$ and define $g(x) = \int_x^a t^{-1} f(t) \, dt$. Show that

$$\int_0^a g(x) \, dx = \int_0^a f(x) \, dx.$$

(3) Let F be increasing on $\mathbb{R}$. Show that for all $a < b$,

$$F(b) - F(a) \geq \int_a^b F'(t) \, dt.$$

(4) Fix a measure space $(X, \mathcal{M}, \mu)$. Suppose $0 < p < q < \infty$ and $L^p(X, \mu)$ is not a subset of $L^q(X, \mu)$. Show that X contains sets of arbitrarily small positive measures.

(5) Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n, m)$, where m is the Lebesgue measure. Show that

$$\lim_{h \to 0} \|f(x+h) - f(x)\|_p = 0.$$

[Hint: first prove this for compactly supported continuous functions.]

Part II. Functional Analysis

Note: You may use any (consistent) normalization that you prefer for Fourier transforms and Fourier series.

(1) Prove or disprove:
   (a) There exists a bounded linear functional $\Lambda : L^\infty([-1,1]) \to \mathbb{R}$ such that $\Lambda u = u(0)$ for all u bounded and continous at 0.
   (b) There exists a bounded linear functional $\Lambda : L^\infty([-1,1]) \to \mathbb{R}$ such that $\Lambda u = u'(0)$ for all u bounded and differentiable at 0.

(2) Let $K \in L^1(\mathbb{R})$. Show that the linear transformation T given by

$$(Tf)(x) = \int_{-\infty}^{\infty} K(x-y) f(y)\, dy$$

is bounded on $L^2(\mathbb{R})$, but is not compact unless $K = 0$. [Hint for the latter part: first show that T commutes with translations.]

(3) Suppose that f is a Schwartz function on $\mathbb{R}$ with

$$\int_{-\infty}^{\infty} x^k f(x)\, dx = 0$$

for all $k \in \{0\} \cup \mathbb{N}$. Is f the zero function? Prove or give a counterexample.

(4) Let X be a Banach space and $S \subset \mathcal{L}(X,X)$ denote the set of invertible bounded linear operators.
   (a) Show that S is open in the operator norm topology.
   (b) Suppose $A \in S$ and $A_j \in \mathcal{L}(X,X)$ with $A_j(x) \to A(x)$ for all $x \in X$ ("strong convergence of operators"). Must there exist j_0 such that $A_j \in S$ for all $j \geq j_0$? Prove or give a counterexample.

(5) (a) State the definition of the Sobolev spaces $H^s(\mathbb{R}^n)$ for $s \geq 0$.
   (b) Let $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ denote the Laplace operator. Show that if $u \in L^2(\mathbb{R}^n)$, $\Delta u = f$ (in the weak sense, a.k.a. in the distributional sense) and $f \in H^s(\mathbb{R}^n)$ ($s > 0$), then $u \in H^{s+2}(\mathbb{R}^n)$ ("elliptic regularity").

Part III. Complex Analysis

(1) Let $f : D \to D$ be holomorphic, where D is the unit disk. Suppose that $f(a) = 0$ for some $a \in D$. Show that

$$|f(z)| \leq \left| \frac{z-a}{1-\bar{a}z} \right|.$$

(2) Let $f : D \setminus \{0\} \to \mathbb{H}$ be holomorphic, where D is the unit disk and $\mathbb{H}$ is the upper half plane. Show that 0 is a removable singularity of f.

(3) Compute the integral

$$\int_0^\infty \frac{1}{(x^2+1)^2}\, dx,$$

using contour integration. Justify any limits that you take.

(4) For $r_1, r_2 \geq 0$ define the annulus $A_{r_1,r_2} = \{z : r_1 < |z| < r_2\}$. Show that there is no biholomorphism $f : A_{1,2} \to A_{1,4}$. You may assume that every such biholomorphism can be extended to a homeomorphism between the two closed annuli.

(5) Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, with simple zeros at $1, i, 2$, and no other zeros. Find

$$\int_{\partial D(0,3/2)} z^2 e^{-z} \frac{f'(z)}{f(z)}\, dz,$$

where $\partial D(0, 3/2)$ is the circle of radius $3/2$ around the origin.