NORTHWESTERN UNIVERSITY
SECOND SELECTION TEST

Thursday, October 7, 2004

NAME: ______________________________

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem B1. Prove that there are no rational numbers u, v, w such that $u^2 + v^2 + w^2 = 7$.

NAME: ____________________

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem B2. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers. Show that for all positive x,

$$(x+a_1)(x+a_2)\ldots(x+a_n) \le \left(x + \frac{a_1+a_2+\cdots+a_n}{n}\right)^n .$$

NAME: ______________________________

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem B3. Let m be an odd positive integer. Prove that there is a positive integer n such that $2^n - 1$ is divisible by m.

NAME: ______________________________

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem B4.

(1) In a 120×150 rectangle (made out of unit squares joined along their sides), how many unit squares does its diagonal pass through?
(2) In a $120 \times 150 \times 180$ cuboid (made out of unit cubes joined along their faces), how many unit cubes does its diagonal pass through?
(Just "touching" at one point does not qualify as passing through).

NAME: ____

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem B5. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and the product of any three elements of U is in U, show that at least one of the subsets T, U is closed under multiplication.

NAME: ______________________________

WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem B6. For positive integers n, define S_n to be the minimum value of the sum

$$\sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2}\,,$$

as the $a_1, a_2, \ldots, a_n$ range through all positive real values such that

$$a_1 + a_2 + \cdots + a_n = 17\,.$$

Find S_{10}.