NORTHWESTERN UNIVERSITY
FIRST TRAINING/SELECTION TEST

Tuesday, May 24, 2005

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WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem A1. Determine the sum of the coefficients of the polynomial that is obtained if we expand the expression $(1 + x - 3x^2)^{2005}$ and reduce like terms.

Answer:

The sum of the coefficients of a polynomial $p(x)$ is just $p(1)$, so the answer is the result of replazing x with 1 in $(1 + x - 3x^2)^{2005}$:

$$(1 + 1 - 3)^{2005} = (-1)^{2005} = \boxed{-1}.$$

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Problem A2. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$, and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that F_{10^k} is odd for every $k \geq 0$.

Answer:

We have that F_n is even precisely when F_{n-1} and F_{n-2} have the same parity, because the sum of two even numbers or two odd numbers is even. If F_{n-1} and F_{n-2} have different parity then F_n will be odd. Consequently the sequence of parities of F_n is of the form *even*, *odd*, *odd*, *even*, *odd*, *odd*, ... with period 3. Since $10^k \equiv 1 \pmod{3}$, F_{10^k} will have the same parity as F_1, i.e., *odd*.

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Problem A3. Find the product $\displaystyle\prod_{n=2}^{2005}\left(1-\frac{1}{n^2}\right)$.

Answer:

For each natural number we have $1-\frac{1}{n^2}=\frac{(n-1)(n+1)}{n^2}$. Hence the product telescopes:

$$\begin{aligned}\prod_{n=2}^{2005}\left(1-\frac{1}{n^2}\right) &= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)\\ &= \frac{1\cdot 3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdot\frac{3\cdot 5}{4^2}\cdots\frac{2003\cdot 2005}{2004^2}\cdot\frac{2004\cdot 2006}{2005^2}\\ &= \frac{1}{2}\cdot\frac{2006}{2005}=\boxed{\frac{1003}{2005}}.\end{aligned}$$

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Problem A4. Find the minimum value of $f(x,y,z) = \sqrt{x+y+z}$ subject to the constrains $x, y, z > 0$, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

Answer:

By the Harmonic Mean-Arithmetic Mean inequality we have:

$$3 = \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \leq \frac{x+y+z}{3} = \frac{1}{3} f(x,y,z)^2 \,.$$

Hence $f(x,y,x) \geq \sqrt{9} = 3$. On the other hand $f(3,3,3) = 3$, so the minimum is $\boxed{3}$.

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Problem A5. Find the value of the following infinite tower of exponents:

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}}$$

interpreted as the limit of the infinite sequence 1, $\sqrt{2}$, $\sqrt{2}^{\sqrt{2}}$, $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$, ...,

Answer:

In this problem the "difficult" part is to prove that the given sequence has a limit x, after that we easily see that the limit must verify the equation

$$x = 2^{x/2}\,,$$

and from here we find that the limit is $x = 2$.

Here is a detailed proof:

The problem asks for the limit of the sequence recursively defined $x_0 = 1$, $x_{n+1} = (\sqrt{2})^{x_n}$ for $n \geq 0$. So first we must make sure that the sequence has a limit. To that end we use the *Monotonic Sequence Theorem*: every bounded monotonic (e.g., increasing) sequence is convergent.

So we first prove that the sequence is *bounded*, more specifically $0 < x_n < 2$ for every n. By induction, the base case is $0 < x_0 = 1 < 2$. Next assume that $0 < x_n < 2$, then $0 < x_{n+1} = 2^{x_n/2} < 2^1 = 2$.

Next we prove that it is *increasing*, i.e., $x_n < x_{n+1}$ for every n. We can do it also by induction. The base case is $x_0 = 1 < \sqrt{2} = x_2$. Next assume $x_n < x_{n+1}$, i.e., $x_{n+1} - x_n > 0$. Then $x_{n+2}/x_{n+1} = 2^{x_{n+1}}/2^{x_n} = 2^{x_{n+1}-x_n} > 1$, hence $x_{n+1} < x_{n+2}$.

So, by the MST the sequence has a limit. Call it x. Then taking limits as $n \to \infty$ in $x_{n+1} = 2^{x_n/2}$ we get $x = 2^{x/2}$, satisfied by $x = 2$. Since the limit is unique, this must be the answer:

$$\lim_{n\to\infty} x_n = \boxed{2}\,.$$

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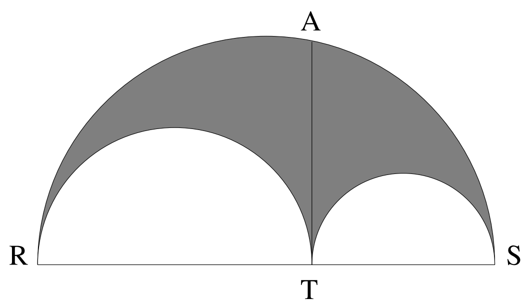
Problem A6. $\overline{RS}$ is the diameter of a semicircle. Two smaller semicircles, $\widehat{RT}$ and $\widehat{TS}$, are drawn on $\overline{RS}$, and their common internal tangent $\overline{AT}$ intersects the large semicircle at A, as shown in the figure. Find the ratio of the area of a semicircle with radius $\overline{AT}$ to the area of the shaded region.



Answer:

If d is the diameter of a semicircle, then its area is $\frac{1}{2}\pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{8}d^2$.

Next, we have $\overline{RS}^2 = (\overline{RT} + \overline{TS})^2 = \overline{RT}^2 + \overline{TS}^2 + 2\overline{RT}\cdot\overline{TS}$, hence the area of the shaded region is $\pi/8$ multiplied by $\overline{RS}^2 - \overline{RT}^2 - \overline{TS}^2 = 2\overline{RT}\cdot\overline{TS}$, i.e., $\frac{\pi}{4}\overline{RT}\cdot\overline{TS}$. The area of a semicircle with radius $\overline{AT}$ is $\frac{\pi}{2}\overline{AT}^2$, hence their ratio is $2\overline{RT}\cdot\overline{TS}/\overline{AT}^2$.

Finally, we use the well known fact that $\overline{AT}^2 = \overline{RT}\cdot\overline{TS}$,[1] hence the ratio is $2\overline{RT}\cdot\overline{TS}/\overline{AT}^2 = 2\overline{AT}^2/\overline{AT}^2 = \boxed{2}$.

[1] Draw $\overline{RA}$ and $\overline{SA}$. In the right triangle RAS, $\overline{AT}$ is perpendicular to $\overline{RS}$. The measure of the altitude on the hypotenuse of a right triangle is the mean proportional between the measures of the segments of the hypotenuse. Hence $\overline{AT}^2 = \overline{RT}\cdot\overline{TS}$.