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2006 NU PUTNAM TEAM SPRING COMPETITION

Problem A1. Find the sum of the digits of

$$9 + 99 + 999 + \cdots + \overbrace{99\ldots9}^{99}\,.$$

- *Answer:* The given number can be written

$$\begin{aligned}(10-1)+(10^2-1)+\cdots+(10^{99}-1) &= \overbrace{11\ldots11}^{99}0 - 99\\ &= \overbrace{11\ldots1}^{97}011\,.\end{aligned}$$

Its digits are 99 ones and once zero, so the sum is 99.

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Problem A2. Let a, b two positive real numbers. Prove that $a^a + b^b \geq a^b + b^a$.

- *Answer:* Assume without loss of generality that $a \leq b$.

There are two cases.

(1) Suppose $b \geq 1$.

We want to show

$$a^b - a^a \leq b^b - b^a\,.$$

This is equivalent to

$$\int_a^b \log a \cdot a^x \, dx \leq \int_a^b \log b \cdot b^x \, dx\,.$$

This is implied by $\log a \cdot a^x \leq \log b \cdot b^x$. If $\log b \geq 0$ we have

$$\log a \cdot a^x \leq \log b \cdot a^x \leq \log b \cdot b^x\,,$$

hence the result.

(2) Suppose $0 \leq a \leq b \leq 1$.

Now we want to show

$$b^a - a^a \leq b^b - a^b\,.$$

This is equivalent to

$$\int_a^b a x^{a-1} \, dx \leq \int_a^b b x^{b-1} \, dx\,.$$

It is enough to show $ax^{a-1} \leq bx^{b-1}$ for $0 \leq a \leq x \leq b \leq 1$. This is equivalent to

$$\log \frac{1}{x} \leq \frac{\log b - \log a}{b - a}\,,$$

which is true because

$$\log \frac{1}{x} \leq \log \frac{1}{a} \leq \frac{-\log a}{1 - a} \leq \frac{\log b - \log a}{b - a}\,.$$

The last inequality is a consequence of concavity of log:

$$\log b = \log \left(\frac{1-b}{1-a} a + \frac{b-a}{1-a} \right) \geq \frac{(1-b)\log a}{1-a} \qquad \Longrightarrow$$

$$\log b - \log b \geq \frac{(1-b)\log a - (1-a)\log a}{1-a} = \frac{-(b-a)\log a}{1-a}\,.$$

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Problem A3. Let x, y, z positive real numbers. Find the minimum value of the function

$$f(x,y,z) = \left(x + \frac{1}{y}\right)\left(y + \frac{1}{z}\right)\left(z + \frac{1}{x}\right).$$

- *Answer:* By the Arithmetic Mean-Geometric Mean Inequality:

$$\frac{1}{2}\left(x + \frac{1}{y}\right) \geq \sqrt{x \cdot \frac{1}{y}} = \frac{\sqrt{x}}{\sqrt{y}},$$

and similarly with the other two factors. Hence:

$$\frac{1}{8} f(x,y,z) \geq \frac{\sqrt{xyz}}{\sqrt{xyz}} = 1.$$

Consequently:

$$f(x,y,z) \geq 8.$$

The lower bound is attained at $(x,y,z) = (1,1,1)$, hence the minimum is 8.

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Problem A4. Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

- *Answer:* Suppose on the contrary that there exist $t_1, t_2 \in T$ with $t_1 t_2 \in U$ and $u_1, u_2 \in U$ with $u_1 u_2 \in T$. Then $(t_1 t_2) u_1 u_2 \in U$ while $t_1 t_2 (u_1 u_2) \in T$, contradiction.

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Problem A5. Prove that if we select $n + 1$ numbers from the set $S = \{1, 2, 3, \ldots, 2n\}$, among the numbers selected there are two such that one is a multiple of the other one.

- *Answer:* For each odd number $2k - 1$, $k = 1, \ldots, n$, consider the set C_{2k-1} = the elements of S of the form $(2k - 1)2^i$ for some $i \geq 0$. The sets $C_1, C_3, \ldots, C_{2n-1}$ are a classification of S into n classes. By the pigeonhole principle, given $n+1$ elements of S, at least two of them will be in the same class. But any two elements of the same class C_{2k-1} verify that one is a multiple of the other one.

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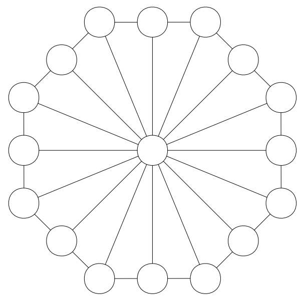
Problem A6. The numbers $1, 2, \ldots, 17$ are placed inside the 17 circles of the figure below, so that the sum along all sides and along all diagonals is the same (note that there are eight sides and eight diagonals.) Prove that there is only one number that can be placed in the center, and find it.



- *Answer:* The number n in the center must be a common element of eight different triples $(n, a+k, b-k)$, $(k = 0, \ldots, 7)$ with the same sum $s = n + a + b$. Let A, B be the sets $A = \{a, a+1, \ldots, a+7\}$, $B = \{b, b-1, \ldots, b-7\}$. A and B can be only the integer intervals $[1, 8]$ and $[10, 17]$, or $[2, 9]$ and $[10, 17]$, or $[1, 8]$ and $[9, 16]$, so n can be only 1, 9 or 17.

If $n = 1$, then the common sum must be 20. The sum of the 16 numbers placed on the circles around the polygon is $2 + \cdots + 17 = 152$. On the other hand each of the eights sides of the polygon must have sum 20. Multiplying by 8 sides we get 160. The difference $160 - 152 = 8$ must be the sum of the elements in the 8 vertices of the polygon, but that is impossible because those elements are grater than 1, and their sum must be greater than 8.

An entirely analogous reasoning allows us to rule out the case $n = 17$. If $n = 17$ then the common sum must be 34. The sum of the 16 numbers placed on the circles around the polygon would be now $1 + \cdots + 16 = 136$. Each side of the polygon must have sum 34, which times 8 yields 272. The difference $272 - 136 = 136$ is the sum of the elements in the vertices of the polygon, but that is impossible, because those elements are less than 17, and their sum must be less than $17 \times 8 = 136$.

So, the only possibility for the number in the center is 9 (and the common sum is 27.)

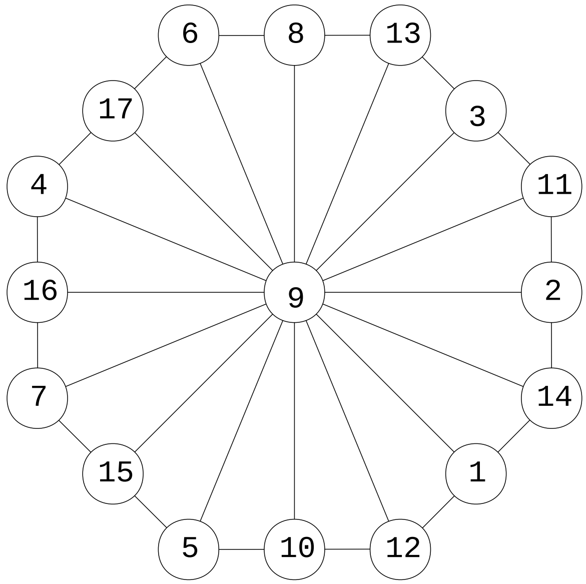


FIGURE 1. A possible solution to the puzzle.