NORTHWESTERN UNIVERSITY Wednesday, Oct 6th, 2010

NAME: ANSWERS

FALL 2006 NU PUTNAM SELECTION TEST

Problem A1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

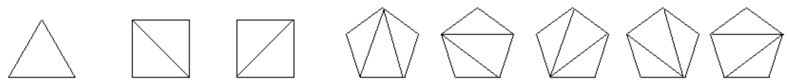
- *Answer:* The function $g(x) = f(x, 0)$ works. Substituting $(x, y, z) = (0, 0, 0)$ into the given functional equation yields $f(0, 0) = 0$, whence substituting $(x, y, z) = (x, 0, 0)$ yields $f(x, 0) + f(0, x) = 0$. Finally, substituting $(x, y, z) = (x, y, 0)$ yields $f(x, y) = -f(y, 0) - f(0, x) = g(x) - g(y)$.

ANSWERS

NAME: ______________________________

FALL 2006 NU PUTNAM SELECTION TEST

Problem A2. A convex polygon of n sides in the plane $\mathbb{R}^2$ can be divided into non-overlapping triangles by non-intersecting line segments connecting pairs of vertices. Suppose that C_n is the number of such divisions, e.g., $C_3 = 1$, $C_4 = 2$, $C_5 = 5$ (see figure).

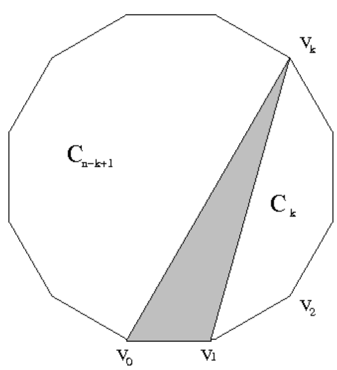


Show that it satisfies the recursive relation

$$C_n = \sum_{k=2}^{n-1} C_k C_{n-k+1} \,,$$

with the convention that $C_2 = 1$.

- *Answer:* Assume the given polygon has vertices $v_0, v_1, \ldots, v_{n-1}$, and let N_k be the number of divisions containing the triangle with vertices v_0, v_1, v_k $(k = 2, \ldots, n-1)$. To each side of that triangle we have polygons with vertices $v_1, v_2, \ldots, v_k$ and $v_k, v_{k+1}, \ldots, v_{n-1}$ respectively. Each can be divided into C_k and C_{n-k+1} ways respectively, yielding $N_k = C_k C_{n-k+1}$ (note that this is true also for the special cases $k = 2$ and $k = n-1$ under the convention $C_2 = 1$). After adding for $k = 2, \ldots, n-1$ we get the desired result.



NAME: ANSWERS

FALL 2006 NU PUTNAM SELECTION TEST

Problem A3. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1^3 + 2^3 + \cdots + n^3}}\,.$$

Hint: What is the closed form for the sum $1^3 + 2^3 + \cdots + n^3$?

- *Answer:* We have that $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$, hence

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1^3 + 2^3 + \cdots + n^3}} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 2\,.$$

ANSWERS

NAME: ______________________________

FALL 2006 NU PUTNAM SELECTION TEST

Problem A4. Let $p(x) = ax^2 + bx + c$ be a polynomials with real coefficients. Assume that for some real number λ, the values $p(\lambda)$, $p(\lambda+1)$, and $p(\lambda+2)$ are integers. Show that $2a$, $2b$, and $2c$ are also integers.

- *Answer:* By hypothesis we have

$$\begin{cases} a\lambda^2 & + & b\lambda & + & c & = & u \\ a(\lambda+1)^2 & + & b(\lambda+1) & + & c & = & v \\ a(\lambda+2)^2 & + & b(\lambda+2) & + & c & = & w \end{cases}$$

where u, v, w are integers. Interpreting the above expression as a system of equations with unknowns a, b, c, and solving by Cramer's rule, we get

$$a = \frac{a'}{\det(M)}, \quad b = \frac{b'}{\det(M)}, \quad c = \frac{c'}{\det(M)},$$

where a', b', c' are integers, and $\det(M)$ is the determinant of the coefficient matrix

$$M = \begin{pmatrix} \lambda^2 & \lambda & 1 \\ (\lambda+1)^2 & (\lambda+1) & 1 \\ (\lambda+2)^2 & (\lambda+2) & 1 \end{pmatrix}.$$

M is a Vandermonde matrix, and its determinant is $\det(M) = (\lambda-(\lambda+1))(\lambda-(\lambda+2))((\lambda+1)-(\lambda+2)) = -2$, so all the denominators in the expressions for a, b, c above are -2, and the desired result follows.

NAME: ANSWERS

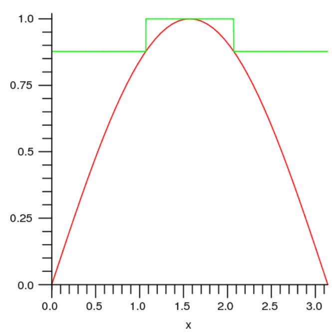
FALL 2006 NU PUTNAM SELECTION TEST

Problem A5. What is the limit

$$\lim_{n\to\infty}\int_0^\pi (\sin x)^n\, dx\,?$$

- *Answer:* Given any $0<\varepsilon<\pi/2$, let f_ε be the function

$$f_\varepsilon(x)=\begin{cases}1 & \text{if } x\in[\frac{\pi}{2}-\varepsilon,\frac{\pi}{2}+\varepsilon]\,,\\ \sin(\frac{\pi}{2}-\varepsilon) & \text{otherwise}\,.\end{cases}$$



Then, we have $0\le\sin x\le f_\varepsilon(x)$ for every $x\in[0,\pi]$, and

$$0<\int_0^\pi(\sin x)^n\,dx\le\int_0^\pi f_\varepsilon(x)^n\,dx=$$
$$\int_0^{\frac{\pi}{2}-\varepsilon} f_\varepsilon(x)^n\,dx+\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} f_\varepsilon(x)^n\,dx+\int_{\frac{\pi}{2}+\varepsilon}^{\pi} f_\varepsilon(x)^n\,dx=$$
$$(\tfrac{\pi}{2}-\varepsilon)\{\sin(\tfrac{\pi}{2}-\varepsilon)\}^n+2\varepsilon+(\tfrac{\pi}{2}-\varepsilon)\{\sin(\tfrac{\pi}{2}-\varepsilon)\}^n\,.$$

Since $0<\sin(\frac{\pi}{2}-\varepsilon)<1$, the first and third term of the last expression tend to zero as $n\to\infty$, hence

$$0\le\lim_{n\to\infty}\int_0^\pi(\sin x)^n\,dx\le 2\varepsilon\,.$$

Since that is true for arbitrary $\varepsilon\in(0,\pi/2)$, it follows that

$$\lim_{n\to\infty}\int_0^\pi(\sin x)^n\,dx=0\,.$$

NAME: ANSWERS

FALL 2006 NU PUTNAM SELECTION TEST

Problem A6. Let $a_1, a_2, \ldots, a_n$ be positive numbers and $b_1, b_2, \ldots, b_n$ be a permutation of this sequence. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$

- *Answer:* Using the Arithmetic Mean-Geometric Mean inequality we get:

$$\frac{1}{n}\left\{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right\} \geq \sqrt[n]{\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdots \frac{a_n}{b_n}} = 1\,.$$

From here the desired result follows.