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Problem A1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

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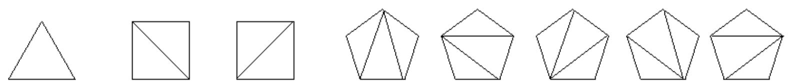
Problem A2. A convex polygon of n sides in the plane $\mathbb{R}^2$ can be divided into non-overlapping triangles by non-intersecting line segments connecting pairs of vertices. Suppose that C_n is the number of such divisions, e.g., $C_3 = 1$, $C_4 = 2$, $C_5 = 5$ (see figure).



Show that it satisfies the recursive relation

$$C_n = \sum_{k=2}^{n-1} C_k C_{n-k+1} \,,$$

with the convention that $C_2 = 1$.

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Problem A3. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1^3 + 2^3 + \cdots + n^3}}\,.$$

Hint: What is the closed form for the sum $1^3 + 2^3 + \cdots + n^3$?

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Problem A4. Let $p(x) = ax^2 + bx + c$ be a polynomials with real coefficients. Assume that for some real number λ, the values $p(\lambda)$, $p(\lambda + 1)$, and $p(\lambda + 2)$ are integers. Show that $2a$, $2b$, and $2c$ are also integers.

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Problem A5. What is the limit

$$\lim_{n\to\infty} \int_0^{\pi} (\sin x)^n \, dx \,?$$

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Problem A6. Let $a_1, a_2, \ldots, a_n$ be positive numbers and $b_1, b_2, \ldots, b_n$ be a permutation of this sequence. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$